Hahn Valuations and (Locally) Compact Rings

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1. Introduction

Valuation theoretic methods play an important role in the study of ordered groups and ordered rings. ([6], [7], [8] and [10]). The archimedean classes of a (fully) ordered ring (0 included) form an ordered semigroup in a natural way. This semigroup which we call a Hahn semigroup enjoys a special property: if \( 0 \neq st = st' \) then \( t = t' \) and similarly if \( 0 \neq ts = t's \) then \( t = t' \). The map which takes an element of the ring into its archimedean class enjoys properties similar to a valuation. This map we call a Hahn valuation. Thus the domain of a Hahn valuation may admit non-trivial zero divisors. Our notion extends the generalizations of a valuation made in [9] and [18].

The Hahn valuation gives rise to a natural topology on the ring \( A \). In this Hahn topology, \( A \) is a topological group. It need not be a topological ring. However, following valuation theory there is a subring \( A^* \) of \( A \)—the Hahn valuation ring—which is always a topological ring with the induced Hahn valuation. A characterization of the compact ring \( A^* \) is presented in Theorem (5.7). If \( K \) is a complete discrete valued field of rank 1 with finite residue class field and \( A^* \) its valuation ring then \( A^* \) is compact and \( K \) is a locally compact valued field. The converse is also true. Following this, we ask: If \( A \) is a Hahn valued topological domain and \( A^* \) its Hahn valuation ring which we assume to be compact, is it possible that \( A \) properly contains \( A^* \)? The surprising answer is that \( A \) must be a complete discrete valued field of rank 1 with finite residue class field and \( A^* \) its valuation ring. The Hahn valuation becomes a field valuation. Our method of proof enables us to obtain a characterization of discrete valuation rings of rank 1 in terms of a Hahn valuation. This is presented first in Theorem (6.4). Theorem (6.7) is the main investigation which gave rise to this paper.

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2. HAHN VALUATIONS

(2.1) Throughout $A$ will denote an associative and distributive ring with identity 1. $S \cup \{0\}$ will be a Hahn semigroup in the sense of (2.2). $S$ is the set of non-zero elements of the semigroup and $S$ is a Hahn semigroup if $S \cup \{0\}$ admits only the trivial zero divisor. All semigroups have an identity 1 which we assume to be greater than 0 (see (2.3) in [2]). When circumstances require it, elements of $S$ may be denoted by $a, \bar{x}$ etc. In that case $\bar{a}, \bar{x}$ etc. stand for values of elements $a, x$ etc. of $A$.

(2.2) DEFINITIONS. a) A Hahn semigroup is a fully ordered semigroup which satisfies the following weak concellative laws:

(i) if $0 \neq st = st'$, then $t = t'$ ($s, t, t' \in S$)
(ii) if $0 \neq ts = t's$, then $t = t'$ ($s, t, t' \in S$).

b) A Hahn valuation is a map $\sigma : A \to S \cup \{0\}$ satisfying the following properties:

(i) $\sigma$ is onto
(ii) $\sigma(a) = 0$ iff $a = 0$ in $A$
(iii) $\sigma(a - b) \leq \max(\sigma(a), \sigma(b))$ for all $a, b \in A$
(iv) $\sigma$ is multiplicative: $\sigma(ab) = \sigma(a) \sigma(b) \forall a, b \in A$.

The triple $(A, S, \sigma)$ is a Hahn valued ring or a ring with a Hahn valuation. When $A$ is a field we speak of a field with a valuation or simply a field valuation.

Throughout $A$ will be a Hahn valued ring with $\sigma$ as the corresponding Hahn valuation. $\sigma(A) = S \cup \{0\}$.

(2.3) SOME EASY DEDUCTIONS. (a) $\sigma(1) = 1_s$ for if $\sigma(1) = \bar{a}$ then $ax = \sigma(1) \sigma(x) = \sigma(x) = \bar{x}$ for every $\bar{x} \in S$; similarly $xa = \bar{x}$ for every $\bar{x} \in S$. Hence $\bar{a} = 1_s$.

b) $\sigma(-a) = \sigma(a)$ for every $a \in A$.

c) $\sigma(a + b) = \sigma(a - b)$ for every $a, b \in A$.

d) A admits no zero divisors if and only if $S \cup \{0\}$ does, that is if and only if $S$ is a semigroup by itself.

c) If $A$ is a division ring then $S$ is an ordered group. Theorem (3.3) shows that the converse may not be true.

(2.4) EXAMPLES OF HAHN VALUATIONS

a) Any non-archimedian absolute value of a division ring is a Hahn valuation.
b) Fully ordered rings furnish a major source of examples. Let $A$ be a fully ordered ring and $S \cup \{0\}$ the set of all archimedean classes of $A$. ($a$ and $b$ are in the same archimedean class if there exist integers $n$ and $m$ such that $|na| \geq |b|$ and $|mb| \geq |a|$). $S \cup \{0\}$ is made into a semigroup in the natural way—define $a \cdot b = \bar{ab}$, the class containing $ab$. Define an order relation in $S \cup \{0\}$ by $\bar{a} < \bar{b}$ if $|a|$ is infinitely smaller than $|b|$, that is $|n_1a| < |b|$ for every integer $n$. We have $0 < S$ and $S \cup \{0\}$ is a fully ordered semigroup with an identity element. Now define $\sigma : A \to S \cup \{0\}$ by $\sigma(a) = \bar{a}$, the archimedean class containing $a$. $\sigma$ is a Hahn valuation, called the order Hahn valuation of $A$.

c) A division ring admits several valuations. A ring may admit several Hahn valuations with the same semigroup of values; for example the ring of integers as a subring of the valued field of rationals. Another interesting example is the following: let $A = \mathbb{Z}[X]$. We consider the anti-lexical and lexical orders of $\mathbb{Z}[X]$. The corresponding two order Hahn valuations are called the anti-lexical and lexical Hahn valuations of $\mathbb{Z}[X]$. In the first case the ordered semigroup of values is given by

$$0 < \cdots < X^n < X^{n-1} < \cdots < X^2 < X < 1$$

and in the other case by $0 < 1 < X < X^2 < \cdots < X^n \cdots$. The following two examples are instructive for the purposes of this paper.

d) Let $K$ be a field and consider the ring of formal power series $A = K[[X, Y]]$ in two variables $X, Y$. An element $a \neq 0$ of $A$ will be written as follows:

$$a = k_{00} + k_{10}X + k_{01}Y + k_{20}X^2 + k_{02}XY + k_{00}Y^2 + \cdots + k_{n0}X^n$$

$$+ k_{n-1, 1}X^{n-1}Y + \cdots + k_{1n-1}XY^{n-1} + k_{0n}Y^n + \cdots .$$

Let $S$ be the free commutative semigroup generated by $X$ and $Y$. We order $S$ as follows—$X^mY^n \leq X^\lambda Y^\mu$ ($m, n, \lambda, \mu$ are non-negative) if $m + n > \lambda + \mu$ or $m + n = \lambda + \mu$ and $m \geq \lambda$. This defines a total order on $S$ and $S$ is a Hahn semigroup. Define a Hahn valuation of $A$ by $\sigma(o) = o$ and if $a \neq o$ then $\sigma(a) = \frac{1}{2}$ the first monomial $X^mY^n$ in the above expression for $a$ which has a non-zero coefficient of $K$. This map will be called the least total degree Hahn valuation of $A$.

e) Let $K$ be a field and $A = K[[\{X_i\}]]$ the ring of formal power series over a field in a countable number of indeterminates $X_i$, $i = 1, 2, 3, \ldots$. Let $S$ be the free commutative semigroup generated by $\{X_i\}_i$. We order $S$ as follows—$1 \geq S$; if $X_{\lambda_1}^{t_1} \cdots X_{\lambda_n}^{t_n}, X_{\mu_1}^{t_1} \cdots X_{\mu_m}^{t_m}$ are two distinct elements with $\lambda_s, \mu_t > 0$ ($s = 1, 2 \cdots n; t = 1, 2 \cdots m$) then we assume $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_m$ and define $X_{i_1}^{t_1} \cdots X_{i_n}^{t_n} < X_{j_1}^{t_1} \cdots X_{j_m}^{t_m}$.
if \( i_1 + \lambda_1 + i_2 + \lambda_2 + i_n + \lambda_n > j_1 + \mu_1 + j_2 + \mu_2 + \cdots + j_m + \mu_m \) or \( \Sigma(i + \lambda) = \Sigma(j + \mu) \) and \( i_1 > j_1 \) or \( \Sigma(i + \lambda) = \Sigma(j + \mu) \), \( i_1 = j_1 \) and \( \lambda_2 > \mu_1 \) and so on and finally if all the indices on the right occur on the left in the order mentioned then \( n > m \). This defines a total order of \( S \) and \( S \) is an ordered semigroup. An element \( a \neq 0 \) of \( A \) is written as \( \Sigma k_s \) where \( k_s \in K \) and the support of \( a = \{ s \in S : k_s \neq 0 \} \) is anti-wellordered in the order of \( S \). We define \( \sigma(a) \) to be the last element in the support of \( a \) and \( \sigma(0) = 0 \). \( \sigma \) is a Hahn valuation called the index-degree Hahn valuation of \( A \).

(2.5) Remark. Observe that in examples c, d and e above, \( S \) is a cancellative semigroup by itself and that it is either well-ordered or anti-well ordered and is of type \( \omega \).

3. HAHN VALUATION RINGS AND RESIDUE CLASS RINGS

(3.1) Let \( (A, S, \sigma) \) be a Hahn valued ring. Consider the subset \( A^* = \{ a \in A : \sigma(a) < 1 \} \) in \( S \). \( A^* \) is a subring of \( A \). We call \( A^* \) the Hahn valuation ring associated to \( \sigma \). If \( P = \{ a \in A : \sigma(a) < 1 \text{ in } S \} \) then \( P \) is a two-sided ideal of \( A^* \). In fact \( P \) is a prime ideal of \( A^* \) in the sense that if \( P \) contains a product of two ideals of \( A^* \), then \( P \) contains one of them. \( P \) is called the prime ideal associated to the Hahn valuation \( \sigma \). As usual, the quotient ring \( A^*/P \) will be called the residue class ring associated to \( \sigma \). We have

(3.2) Proposition. The residue class ring \( A^*/P \) admits only the trivial zero divisor.

Proof. If \( ab \in P \) and both \( a \) and \( b \) are not in \( P \) then \( \sigma(a) = \sigma(b) = 1 \) from which \( \sigma(ab) = 1 \), a contradiction.

On the existence of rings with Hahn valuations, we have the following result.

(3.3) Theorem. Given a Hahn semigroup \( S \cup \{0\} \) and a ring \( B \) admitting only the trivial zero divisor, there exists a ring \( A \) and a Hahn valuation \( \sigma \) from \( A \) onto \( S \cup \{0\} \) such that the residue class ring associated to \( \sigma \) is isomorphic to the ring \( B \).

Proof. Let \( A \) be the semigroup ring \( B(S \cup \{0\}) \) (see [2]). An element \( a \neq 0 \) of \( A \) is written as a finite sum \( a = \Sigma b_s \) (\( s \in S \)), where each \( b_s \neq 0 \). Define \( \sigma(a) \) to be the maximum of the elements \( s \) occurring in the above expression for \( a \). If \( \sigma(0) = 0 \) then \( \sigma \) is a Hahn valuation onto \( S \cup \{0\} \); we need to check only that \( \sigma \) is multiplicative. If \( a = \Sigma a_s \) with \( \sigma(a) = t_1 \) and \( b = \Sigma b_s \) with \( \sigma(b) = t_1 \) then \( a_{s_1} \neq 0 \) and \( b_{t_1} \neq 0 \). Thus if \( s_1 t_1 \neq 0 \) then
\( \sigma(ab) = s_1 t_1 = \sigma(a) \sigma(b) \), because \( S \cup \{0\} \) is a Hahn semigroup. If on the other hand \( s_1 t_1 = 0 \) then \( st = 0 \) for all \( s \leq s_1, t \leq t_1 \) so that \( ab = 0 \). Once again \( \sigma(ab) = 0 = \sigma(a) \sigma(b) \). It is easy to check that the residue class ring associated to \( \sigma \) is isomorphic to \( B \).

(3.4) **Remark.** The map \( \sigma \) above is the natural Hahn valuation of the ring \( B(S \cup \{0\}) \). Theorem (3.3) gives examples of rings with Hahn valuations admitting non-trivial zero divisors. The following propositions show that we have a situation here reminiscent of fully ordered rings.

(3.5) **Proposition.** Let \( A \) be a ring with a Hahn valuation. If \( ab = 0 \) then \( a^2 = 0 \) or \( b^2 = 0 \).

**Proof.** Assume \( \sigma(a) \leq \sigma(b) \), so that \( 0 \leq \sigma(a^2) = \sigma(a) \sigma(a) \leq \sigma(a) \sigma(b) = \sigma(ab) = 0 \). Hence \( a^2 = 0 \).

Because of the analogy above, it is convenient to make the following definitions.

(3.6) **Definitions.** An ideal \( I \) of the ring \( (A, S, \sigma) \) is a \( \sigma \)-ideal if \( a \in I \), \( b \in A \), \( \sigma(b) \leq \sigma(a) \) imply \( b \in I \). We recall that \( a \in A \) is nilpotent if \( a^n = 0 \) for some integer \( n \). An ideal \( I \) is nilpotent if \( I^n = (0) \) for some \( n \), \( I \) is locally nilpotent if every finitely generated ideal contained in \( I \) is nilpotent and \( I \) is nil if every element of \( I \) is nilpotent. \( I \) is a nil radical of \( A \) if \( I \) is nil and \( A/I \) contains no nonzero nilpotent ideals.

(3.7) **Proposition.** The \( \sigma \)-ideals of \( A \) are totally ordered with respect to inclusion.

**Proof.** If \( I, J \) are \( \sigma \)-ideals and \( I \nsubseteq J \) then there exists \( x \in I \) with \( x \notin J \). If \( y \in J \) then \( \sigma(y) < \sigma(x) \) so that \( y \in I \). Consequently \( J \subseteq I \).

(3.8) **Lemma.** If \( \bar{N}_n = \{ a \in A : a^n = 0 \} \) then \( \bar{N}_n \) is a \( \sigma \)-ideal which is nilpotent.

**Proof.** If \( a \in \bar{N}_n \), \( r \in A \) then \( a^n = 0 \). Let \( \bar{ar} \leq \bar{ra} \). Then \( \bar{r^2a^2} \geq \bar{r(ar)a} = \bar{(ra)^2} \) and by induction \( 0 = \bar{r^n a^n} = \bar{r(r^{n-1}a^{n-1})a} \geq \bar{r(ra)^{n-1}a} \geq \bar{r(ar)^{n-1}a} = \bar{(ra)^n} \geq 0 \) (because 0 is the least element) so that \( (ra)^n = 0 \) and \( ra \in \bar{N}_n \). From \( (\bar{a})^n \leq (\bar{ra})^n \) we have \( ar \in \bar{N}_n \). Also if \( a, b \in \bar{N}_n \) then \( a + b \leq a \) (say) so that \( (a + b)^n \leq a^n = 0 \). Hence \( a + b \in \bar{N}_n \) and \( \bar{N}_n \) is an ideal. It is clear that \( \bar{N}_n \) is a \( \sigma \)-ideal. Also the \( n \)th power of \( \bar{N}_n \) is 0. If \( a_1, a_2, \ldots, a_n \in \bar{N}_n \), let \( \bar{a} = \max. \{ \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \} \). Then \( \sigma(a_1 a_2 a_n) \leq \bar{a}^n = 0 \). Hence \( a_1 a_2 a_n = 0 \). The following is the analogue of Theorem (2.14) in [2].

(3.9) **Theorem.** Let \( (A, S, \sigma) \) be a ring with a Hahn valuation and \( N \) the set of all nilpotent elements in \( A \). Then
(i) $N$ is a prime ideal and a $\sigma$-ideal

(ii) $A/N$ admits only the trivial zero divisor and $A/N$ is a Hahn valued ring in a natural way

(iii) $N$ is a locally nilpotent ideal

(iv) $N$ is the unique nil radical of $A$ and

(v) if $\tau$ is any Hahn valuation of $A$ then $N$ is a $\tau$-ideal.

Proof. The easy proof follows the pattern in [2]. We only expound (ii). $N$ is not only a prime ideal, but has even the property that $ab \in N$ implies $a \in N$ or $b \in N$. Thus $A/N$ admits only the trivial zero divisor. Let $\Sigma = S \setminus \sigma(N)$. $\Sigma$ is a semigroup admitting only the trivial zero divisor. For this note that $\sigma(N)$ is the set of all nilpotent elements of $S$ and is an ideal of $S$. Thus if $a, b \in \Sigma$ then $\sigma(ab) \in \sigma(N)$ would imply that $(ab)^n = 0$. If $a \leq b$ then $a^n \leq ab$ whence $a^{2n} = 0$. So $a \in \sigma(N)$, a contradiction. Thus $\sigma(ab) \in \Sigma$. Now, define a map $\tau : A/N \rightarrow \Sigma \cup \{0\}$ by $\tau(N) = 0$ and $\tau(a + N) = \sigma(a)$ if $a \notin N$. This is well defined for, $\sigma(a) \subset \Sigma$, if $b \divides N = a + N$ then $b - a \in N$. If $\sigma(b) < \sigma(a)$, then $\sigma(b - a) = \sigma(a) \in \Sigma$ so $b - a \notin N$, a contradiction. Similarly $\sigma(a) < \sigma(b)$ and so $\sigma(a) = \sigma(b)$. It is a routine checking that $\tau$ is a Hahn valuation.

(3.10) The idempotents of group rings $A(G)$ where $A$ has a nil ideal $N$ such that $A/N$ admits only the trivial zero-divisor have been studied in [22]. However, one observes that the identity $1$ and $0$ are the only idempotents of the ring $A$; for if $x^2 = x$, then $x(1-x) = 0$. So $x \leq 1$. If $x = \bar{1}$ then $1 - x = 0$ and $x = 1$. If $\bar{x} < 1$ then $\bar{x} = 0$ and $x = 0$.

4. TOPOLOGY DEFINED BY A HAHN VALUATION

(4.1) Given $s \in S$, let $N_s = \{a \in A : \sigma(a) \leq s\}$; $N_s$ is a subgroup of $A$. The family $\mathcal{H} = \{N_s : s \in S\}$ is a family of subgroups fully ordered with respect to inclusion. By (4.21) in [11], it is clear that the family of all sets of the form $a + N$, as $a$ runs through $A$ and $N$ runs through $\mathcal{H}$ is an open basis of $A$ in some topology for $A$ and that with this topology $(A, +)$ is a 0-dimensional topological group. In fact every member of $\mathcal{H}$ is both open and closed. It is clear that $A$ is non-discrete. We call this topology the Hahn topology of $A$. The topology is Hausdorff if and only if $S$ does not have a first element. We next ask as to when $(A, S, \sigma)$ is a topological ring in the Hahn topology. We have:

(4.2) Proposition. $(A, S, \sigma)$ is a topological ring in the Hahn topology
if and only if $S$ satisfies the following condition: given $s, t \in S$, there exist $r, r' \in S$ with $r't \leq s$ and $tr \leq s$ in $S \cup \{0\}$.

Proof. The condition is necessary. Let $\sigma(y) = t$. Since $y \cdot 0 = 0$, using the continuity of multiplication, there exists $N_r$ with $yN_r \subseteq N_s$. It follows that $tr \leq s$. Similarly for the other condition. For sufficiency, we first notice that $r$ and $r'$ in the given condition may be chosen to be less than or equal to $s$; for if $t > 1$ then $ts \leq s$ and $st \leq s$. So $r = r' = s$ will suffice. If $t > 1$, then necessarily $r \leq s$, $r' \leq s$; if not from $r > s$, we will have $tr \geq r > s$, a contradiction. Similarly for $r'$. Now to prove multiplication is continuous. Let $ab = c$ in $A$ and $c + N_s$ a neighbourhood of $c$ with $s \leq 1$. Choose $r \leq s$, $r' \leq s$ in $S$ so that $r'b \leq s$ and $ar \leq s$ in $S \cup \{0\}$. Then $(a + N_r)(b + N_s) \subseteq c + N_s$, since $n \in N_r$, $m \in N_s$ imply that

$$
\sigma(nb + am + nm) \leq \max(r'b, ar, r'r) \leq s.
$$

For rings admitting only the trivial zero divisor, these conditions are nicely simplified.

(4.3) Corollary. If the Hahn valued ring $A$ admits only the trivial zero divisor, then $A$ is a topological ring in the Hahn topology if $S$ satisfies one (and hence both) of the following conditions: (i) given $t > 1$ in $S$ there exists $r' \in S$ with $r't \leq 1$ (ii) given $t > 1$ in $S$, there exist $r \in S$ with $tr \leq 1$ in $S$.

Proof. If $r't \leq 1$, then $r'tr' \leq r'$ and by the cancellative law, $tr' \leq 1$. Thus (i) implies (ii) and conversely. The sufficiency is the only part to be verified. If $r't \leq 1$ and $s \in S$ then $sr't \leq s$ and $tr's \leq s$ and both $sr'$ and $r's$ are not zero, hence in $S$.

(4.4) Remark. The ring $\mathbb{Z}[X]$ with the lexical Hahn valuation is not a topological ring in the Hahn topology. On the other hand example (5.5) shows that for rings admitting non-trivial zero divisors, the conditions of (4.2) can not be improved. However the following result is easily verified.

(4.5) Proposition. Let $A^*$ be the Hahn valuation ring associated to the ring $(A, S, \sigma)$. Then $A^*$ is a topological ring in the Hahn topology. Also each $N_s(s \leq 1)$ is a $\sigma$-ideal of $A^*$.

It is natural to think of the topological completion of the uniform space $A$ in the Hahn topology. In preparation, we have the following results:

(4.6) Lemma. Consider the Hahn topology of $(A, S, \sigma)$. Then (i) if \( \{x_n : n \in D\} \) is a Cauchy net which does not converge to 0 then there exists $d \in D$ so that for every $m \geq d$, $\sigma(x_m) = \sigma(x_d)$ (ii) if a net $\{x_n : n \in D\}$ converges to a point $x \neq 0$ in $A$, then $\sigma(x) = \sigma(x_m)$ for all sufficiently large $m$. 
Proof. Suppose (i) is not true. Then given \( d \in D \) we can always find \( \alpha, \beta \geq d \) in \( D \) so that \( \sigma(x, < \sigma(x,). \) Since \( \{ x_n : n \in D \} \) is a Cauchy net given \( s \in S \), there exists \( d \in D \) such that for all \( \mu, \nu \geq d \), \( x_\mu - x_\nu \in N_s \). Let \( \alpha, \beta \geq d \) such that \( \sigma(x_\alpha) < \sigma(x_\beta) \) and from \( x_\alpha - x_\beta \in N_s \) we have \( \sigma(x_\beta) \leq s \). Then \( \sigma(x_\alpha) \leq s \) for every \( \mu \geq d \). Otherwise \( \sigma(x_\alpha) > s \) and \( \sigma(x_\mu - x_\nu) = \sigma(x_\mu) > s \) — contradicting the fact that \( x_\mu - x_\nu \in N_s \). Thus \( x_\alpha \in N_s \) for every \( \mu \geq d \) so that the Cauchy net \( \{ x_m : m \in D \} \) converges to 0, again contradicting the hypothesis. So (i) holds. Result (ii) follows from the fact that a convergent net is a Cauchy net.

(4.7) Proposition. Consider the Hahn topology of \((A, S, \sigma)\). Then the ideals \( P, N \) are both open and closed in the Hahn topology.

Proof. \( P \) and \( N \) are obviously open—for example \( P = U_{x < 1} N_x \). Suppose \( \{ x_n : n \in D \} \) is a net in \( P \) which converges to \( x \) not in \( P \). So \( x \neq 0 \) and \( \sigma(x) \leq \sigma(x_n) \) for sufficiently large \( n \). But \( \sigma(x_n) < 1 \), a contradiction. So \( P \) is closed. Similarly \( N \) is closed. We now state the theorem on the completion.

(4.8) Theorem. Let \((A, S, \sigma)\) be a Hahn valued ring. Assume that \( A \) is a Hausdorff topological ring in the Hahn topology and consider the uniform completion \( \tilde{A} \) of \( A \) with respect to this topology. Then

(i) \( \tilde{A} \) is a ring and a Hausdorff topological ring in the usual way in the completion topology.

(ii) If \( A \) is commutative, so is \( \tilde{A} \) and if \( A \) admits only the trivial zero divisor so does \( \tilde{A} \).

(iii) \( \tilde{A} \) is a ring with a Hahn valuation \( \hat{\sigma} \) in a natural way and \( \hat{\sigma}(\tilde{A}) = S \). \( \hat{\sigma} \) extends \( \sigma \).

(iv) The Hahn topology of \( \tilde{A} \) given by \( \hat{\sigma} \) is equivalent to the completion topology of \( \tilde{A} \).

(v) With the usual notation, \( N_{\hat{\sigma}(a)} = \tilde{N}_{\sigma(a)} = \text{closure of } N_{\sigma(a)} \) in \( \tilde{A} \).

(vi) The Hahn valuation ring \( (\tilde{A})^* \) associated to \( (\tilde{A}, S, \hat{\sigma}) \) is the closure of \( A^* \) in \( \tilde{A} \) and we have \( (\tilde{A})^* = (\tilde{A})^* = c1 \cdot A^* \) in \( \tilde{A} \).

(vii) If \( P_\sigma \) is the Hahn valuation ideal associated to \( \sigma \), then \( P_\sigma \) is the closure of \( P \) in \( \tilde{A} \) and \( P_\sigma = \tilde{P} \), completion of \( P = \text{closure of } P \) in \( \tilde{A} \).

(viii) If \( N_\sigma \) is the unique nilradical of \( \tilde{A} \) then \( N_\sigma \) is the closure of \( N \) in \( \tilde{A} \) and again we have \( N_\sigma = \tilde{N} \), completion of \( N = c1 \cdot N \).

(ix) The \( \hat{\sigma} \)-ideals of \( \tilde{A} \) are closures of the corresponding \( \sigma \)-ideals of \( A \).

Proof. (i) and part of (ii) are from the well known Proposition 6, p. 79
of [3]. The rest of (ii) follows from (viii), if we observe that \( N = (0) \) if and only if \( A \) admits only the trivial zero divisor. For (iii), we note that an element \( x \neq 0 \) of \( \bar{A} \) is a Cauchy net \( \{x_n : n \in D\} \) of \( A \). So by lemma (4.6), \( \sigma(x_n) \) is constant for sufficiently large \( n \in D \). Define \( \delta(x) \) to be this constant element in \( S \) and \( \delta(0) = 0 \). The results in (iii) are easily checked. To check (iv), it is enough to prove that if \( 0 \neq x = \{x_n : n \in D\} \) is a Cauchy net of \( A \), then this Cauchy net converges to \( x \) in the Hahn topology. Since \( \{x_n : n \in D\} \) is a Cauchy net, given \( s' \in S \), there exists \( d \in D \) such that for \( n, m > d \), \( \sigma(x_n - x_m) \leq s' \). We claim that for \( n \geq d \), \( \sigma(x - x_n) \leq s' \). If not, let \( \delta(x - x_n) = t > s' \) for some \( n \geq d \). Then \( \sigma(x - x_m) = t \) for every \( m \geq d \) since \( \delta(x_n - x_m) \leq s' \). For every \( m \geq d \), we write \( x = x_m + a_m \) with \( \sigma(a_m) = t \). Then \( \{a_m : m \geq d\} \) is a Cauchy net in \( \bar{A} \), which converges to \( a \) say in \( \bar{A} \). So the Cauchy net \( \{x_m + a : m \geq d\} \) converges to both \( x \) and \( x + a \) whence \( a = 0 \). This is a contradiction, as \( a_m = t \) a non-zero element for all \( m \geq d \). Thus \( \sigma(x - x_n) \leq s' \) for every \( n \geq d \). By varying \( s' \), we prove that \( \{x_n : n \in D\} \) converges to \( x \) in the Hahn topology of \( \bar{A} \). Thus the two topologies of \( \bar{A} \) are equivalent. All the other results are easily verified.

5. When is the Hahn Valuation Ring \( A^* \) Compact?

Our investigation shows that it is of interest to know when \( A^* \) is compact. In anticipation of Theorem (5.7), we prove a series of results.

(5.1) Proposition. The prime ideal \( P \) of \( A^* \) may not be a maximal ideal but it is the only maximal \( \sigma \)-ideal of \( A^* \).

Proof. The second part is obvious. For the ring \( \mathbb{Z}[X] \) with the anti-lexical Hahn valuation, \( P \) is the ideal \( (X) \) which is not maximal.

It is natural to consider the localization \( A^*_p \), when \( A^* \) is commutative. \( S^* \cup \{0\} \) is the value semigroup of \( A^* \). For the remainder of this paper \( A, A^* \) are assumed to be commutative unless otherwise stated. Also a local ring is not assumed to be noetherian.

(5.2) Proposition. Let \( (A^*, S^*, \sigma) \) be a Hahn valuation ring and \( P \) its (prime) Hahn valuation ideal. Then the localization \( A^*_p \) is a Hahn valuation ring in a natural way. The Hahn valuation of \( A^*_p \) extends \( \sigma \). Moreover, \( A^* \) is a subring of the local ring \( A^*_p \).

Proof. The last statement comes first. Observe that the complement of \( P \) is \( \{a \in A^* : \bar{a} = 1\} \). This multiplicative set does not admit any zero divisor of \( A^* \). Hence the natural map \( \iota : A^* \to A^*_p \) is a monomorphism and thus \( A^* \) may be considered as a subring of \( A^*_p \). Define \( \sigma_P \) from \( A^*_p \) to \( S^* \cup \{0\} \) by
\( \sigma_P(a/m) = \sigma(a)(a \in A^*, m \notin P) \). \( \sigma_P \) is well-defined and \((A_P^*, S^*, \sigma_P)\) is a Hahn valuation ring (because \( S^* \leq 1 \)). Obviously \( \sigma_P \) extends \( \sigma \). The other conclusions are easily checked.

It is of interest to know when \( A^* \) will be a local ring, with \( P \) as its maximal ideal.

Of course, it is necessary and sufficient that every element \( a \) such that \( \sigma(a) = 1 \) be a unit of \( A^* \). Also

(5.3) Proposition. Let \( S^* \) be archimedean (in the sense of (5.4) below), \( A^*/P \) be a field and \( A^* \) be complete in the Hahn topology. Then \( A^* \) is a local ring and \( P \) its maximal ideal.

Proof. We show that every element not in \( P \) is a unit. If \( \sigma(a) = 1 \), then \( a + P \neq 0 \) since \( A^*/P \) is a field, there exists \( b \in A^* \) with \((a + P)(b + P) = 1 + P \). From this \( ab - 1 \in P \) so that \( ab - 1 = x \) say with \( x \in P \). Consider the sequence of elements \((s_n)\) where \( s_n = 1 + x + x^2 + \cdots + x^{n-1}, \ n = 1, 2, \ldots \). This sequence \((s_n)\) is a Cauchy sequence, for given \( t \in S^* \) which is archimedean, there exists \( m \) an integer so that \( \sigma(x)^m < t \). Now for every \( n \geq m, s_n - s_m = x^n + x^{n-1} + \cdots + x^m \) and \( \sigma(s_n - s_m) \leq \sigma(x^m) \leq t \). Since \( A^* \) is complete \((s_n)\) converges to a point \( s \). Also \( s(1 - x) = \lim s_n(1 - x) \) \((A^* \) is a topological ring\) = 1. Thus \((1 - x)\) is a unit. Hence \( a \) is a unit and \( A^* \) is local.

(5.4) Definition. Let \( S^* \cup \{0\} \) with \( 0 < S^* \leq 1 \) be a fully ordered semigroup. We say that \( S^* \) is archimedean if \( r < 1, t \in S^* \implies r^m \leq t \) in \( S^* \cup \{0\} \) for some integer \( m \).

(5.5) Examples. Most of the examples in (2.4) are archimedean. The following example brings the distinction between the usage of \( S^* \) and \( S^* \cup \{0\} \). Let \( K \) be a field and \( X_1, X_2, \ldots, X_n \) a countable number of indeterminates. \( S^* \cup \{0\} \) is the Hahn semigroup given by

\[
0 < \cdots < X_n < \cdots < X_2 < X_1 < 1; \ X_iX_j = 0 \quad \text{for all } i, j.
\]

Consider the semigroup ring \( K[S^* \cup \{0\}] \) with the natural Hahn valuation. Its completion in the Hahn topology is a local ring. \( S^* \) is archimedean in the above sense.

(5.6) We now state our theorem on the compactness of \( A^* \). This generalizes the well-known result about compact valuation rings of rank 1. Our proof uses the well-known result in Kelley p. 198 [17]—A uniform space is compact relative to the uniform topology if and only if it is complete and totally bounded. \( A^* \) is totally bounded in the Hahn topology if and only
if given any neighborhood $N_0$ of 0, there exists a finite subset $F_s$ of $A^*$ so that $x \in A^*$ implies $x - f \in N_s$ for some $f \in F_s$.

(5.7) Theorem. Let $(A^*, S^*, \sigma)$ be a Hahn valuation ring which is not necessarily commutative. Then $A^*$ is Hausdorff compact in the Hahn topology if and only if (i) $A^*$ is complete (ii) $S^*$ is anti-well ordered and is of type $\omega$ (iii) the quotient of two successive groups $N_s, N_{s+1}(N_{s+1} \subseteq N_s)$ is a finite group.

Proof. If $A^*$ is compact, then it is complete and totally bounded. To prove (ii), we show that given any element $y \in S^*$, there are only a finite number of elements of $S^*$ which exceed $y$. If not let \{\bar{x}_n\} be an infinite sequence of distinct elements of $S^*$ all exceeding $y$. Corresponding to each $\bar{x}_n$ we choose $x_n \in A^*$ so that $\sigma(x_n) = \bar{x}_n$. The sequence $(x_n)$ of $A^*$ contains all distinct elements. Consider the neighborhood $N_y$. There exists a finite subset $F_y = \{f_1, f_2, \ldots, f_m\}$ of $A^*$ so that $a \in A^*$ implies $a - f \in N_y$ for some $f \in F_y$. Since $(x_n)$ is a sequence of distinct elements, there exist distinct indices $i, j$ so that $x_i - f \in N_y$ and $x_j - f \in N_y$ for the same $f \in F_y$. This means $x_i - x_j \in N_y$ which is however impossible, as

$$\sigma(x_i - x_j) = \max(\bar{x}_i, \bar{x}_j) > y.$$  

This proves (ii). Let $S^*$ be given by \( \cdots < s_n < \cdots < s_0 < s_1 < 1 \). Thus the quotient of two successive groups $N_s, N_{s+1}$ makes sense. Observe that $N_s$ is closed and hence compact. Thus $N_s/N_{s+1}$ is a finite group since $N_s$ is totally bounded and $N_{s+1}$ a neighborhood of 0 in $N_s$. For the converse, we need only to prove that $A^*$ is totally bounded. In turn, it is enough to prove that the quotient group $A^*/N_s$ is finite for every $s \in S^*$. First $A^*/N_1$ is finite since $A^* = N_1$. Since $A^*/N_s \simeq (A^*/N_{s+1})/(N_s/N_{s+1})$ induction completes the proof.

(5.8) We now present an example to show that condition (ii) of Theorem (5.7) can not be relaxed without further assumptions—for example Noetherian. Let $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ be a strictly increasing sequence of finite fields. Let $A^*$ be the ring of all formal expressions $k_0 + k_1x + k_2X^2 + \cdots + k_nX^n + \cdots$ where $k_i \in K_i \ (i = 0, 1, 2, 3, \ldots)$. Addition and multiplication are defined in the usual way: $k_iX^i \cdot k_jX^j = k_ik_jX^{i+j}$. Since $K_i$ and $K_j$ are both contained in $K_{i+j}$, $A^*$ is a domain. $S^* = \{\cdots < X^0 < \cdots < X^2 < X < 1\}$ is a semigroup and with the least degree Hahn valuation, $A^*$ is a compact ring. Here $N_s/N_{s+1} \simeq K_s$, $s = 0, 1, 2, \ldots$. If $K_i$ were infinite for some $i \geq 0$, $A^*$ will not be compact.

(5.9) Corollary. If $A^*$ is Hausdorff compact (and commutative) in the Hahn topology, it is a local ring with $P$ as its maximal ideal.
Proof. $S^*$ is anti-well ordered and countable. Hence it is archimedean. The residue class ring $A^*/P$ is a field. Thus the conclusion follows from (5.3).

6. A Characterization of Discrete Valuation Rings of Rank 1

Theorem (5.7) generalizes only part of the results known about compact valuation rings. Here we shall present these known results and give a simple (and possibly new) proof. The usual proofs given in [20] and [5] do not use the (richer) uniform structure of $A^*$.

(6.1) Theorem. Let $K$ be a field with a non-trivial valuation $v$ of rank 1 and $A^*$ its valuation ring. Then the following conditions are equivalent.

1. $A^*$ is compact in the topology defined by $v$.
2. $v$ is a discrete valuation, $K$ is complete with respect to $v$ and $K = A^*/P$ is finite.
3. $K$ is locally compact in the topology defined by $v$.

Proof. We change to the additive notation. Since $A^*$ is totally bounded, it follows as in (5.7) that the additive value semigroup $S^*$ of $A^*$ is well-ordered and infinitely countable. $S^*$ is the positive cone of the ordered group $v(K)$. Thus $v(K)$ is order isomorphic to the additive group of integers. Again (5.7) shows that $K$ is finite. Finally $K$ is complete since $A^*$ is complete, for this if $(x_n)$ is a Cauchy sequence in $K$, then there exists $d$ such that $v(x_n) = v(x_d)$ for all $n \geq d$. If $v(x_d) \geq 0$ then $(x_n)$ is a Cauchy sequence in $A^*$ and hence converges. If $v(x_d) < 0$, then $v(x_n^{d-1}) = 0$ for all $n \geq d$. Thus $(x_n x_d^{-1})_{n \geq d}$ is a Cauchy sequence in $A^*$ ($x_d^{-1} \in A^*$) which converges to $x$ in $A^*$ say. So $(x_n)$ converges to $x$ by continuity of multiplication. (2) implies (1). Since $A^*$ is closed in $K$, it is complete. Since $N_s = P_s$ in this case ($s$ integer) we only have to show that $P^n/P^{n+1}$ is finite. But $P^n/P^{n+1} \simeq A^*/P$ by the mapping $x \in A^* \to xt^n + P^{n+1}$ where $t$ is the element of $A^*$ with $v(t) = 1$, that is $P = At$. The equivalence of (1) and (3) is again an easy checking.

We take the problem of generalizing theorem (6.1) in its entirety. The obvious framework now is to start with a Hahn valued domain $(A, S, o)$. $A^*$ is the corresponding Hahn valuation ring. We assume that $A^*$ is Hausdorff compact in the Hahn topology and is properly contained in $A$. It is natural to assume that $A$ is a topological ring (which it need not be) in the Hahn topology. In this setting it turns out that Theorem (6.1) is the best that can be expected, since $A$ has to be a field and $S$ is an ordered group, order isomorphic to the ordered group of integers. $A^*$ is the valuation ring of the discrete valued field $A$ of rank 1. Since the assumption that $A$ is a topological
ring can be presented as a property of $S$ we first give a few preliminary results, which are of interest in themselves.

(6.2) Proposition. Let $S$ be a commutative Hahn semigroup and $(A, S, \sigma)$ a domain with a Hahn valuation. Then $\sigma$ can be extended to a valuation of the quotient field of $A$.

(6.3) Proposition. Let $S$ be a commutative Hahn semigroup (with an identity element) and let $S^*\{s \in S : s < 1\}$ properly contained in $S$, anti-well ordered and of type $\omega$. Moreover let $S$ satisfy the following property: for every $t \in S$ with $t > 1$ there exists $s \in S$ with $st < 1$. Then $S$ is an ordered group and is order isomorphic to the ordered group of integers.

Proof. We use the hypothesis that there are only a finite number of elements of $S^*$ exceeding a given element. If $Z > 1$, there exists $Y \in S^*$, $X \in S^*$ with $YZ = X$. Assume that there exists no $s \in S^*$ such that $sZ = 1$. We will show that this is impossible. We have $X < 1$ and $Y < X$ and $Y$ is not a power of $X$ ($S^*$ is cancellative). $S^*$ is archimedean and since it is anti-well ordered we have a sequence like this:

$$1 > X > X^2 > \cdots > X^{n_1} > Y > Y^2 > \cdots > Y^{m_1} > X^{n_1+1} > \cdots$$

$$\cdots > X^{n_2} > Y^{m_1+1} > \cdots > Y^{m_2} > \cdots$$

$$\cdots > X^{n_k} > \cdots > Y^{m_k} > X^{n_k+1} > X^{n_k+2} > \cdots$$

where $n_i, m_i > 1$ for $i = 1, 2, \ldots$. The $n_i$ and the $m_i$ are strictly increasing. Also strict inequality holds throughout, since $Y^m = X^n$ is ruled out. For this note that $Y^m = X^n$ and $Y^mZ^m = X^m$ together imply $X^mZ^m = X^m$ so that either $sZ = 1$ for some $s \in A^\ast$ or $Z^m \leq 1$ and these are not the case. Observe that for each $k$, $n_k > m_k$, for $X > Y$ and $X^{n_k} > Y^{m_k} > X^{n_k+1}$. Thus if $n_k < m_k$ then $X^{n_k} > X^{m_k} > Y^{m_k}$ which is impossible. After this, we get

$$Y^{m_1+m_k} \leq Y^{1+m_k} < X^{n_k+1} < Y^{m_k}$$

for $k = 1, 2, 3, \ldots$

so that $Y^{m_1} < X^{n_k-m_k+1}Z^{m_k} < 1$ for every $k = 1, 2, 3, \ldots$. For this $Y^{m_k}Z^{m_k} = X^{m_k}$ and so $X^{n_k-m_k+1}Y^{m_k}Z^{m_k} = X^{n_k+1}$ lies between $Y^{m_1+m_k}$ and $Y^{m_k}$ from which the result follows. On the other hand, all these elements are distinct. This contradicts the hypothesis that there are only a finite number of elements exceeding $Y^{m_1}$. Thus our initial assumption does not hold and there indeed exists $Y$ such that $YZ = 1$ and this is true of every $Z > 1$. We claim that $Y$ is a power of the second largest element $X(<1)$ of $S^*$. We distinguish two cases.
Case 1. Assume \( Y^m = X^n \) is never possible \((YZ = 1)\). We will get a strictly decreasing sequence as before. In this case again

\[
Y^{m_1+m_k} < X^{n_k+1} < Y^{m_k} \quad \text{and} \quad Y^{m_1} < X^{n_k+1}Z^{m_k} < 1
\]

for every \( k = 1, 2, 3, \ldots \).

All these last elements are distinct, for if \( X^{n_k+1}Z^{m_k} = X^{n_i+1}Z^{m_i} \), let \( k > i \), \( X^{n_k-n_i}Z^{m_k-m_i} = 1 \) and so \( X^{n_k-n_i} = Y^{m_k-m_i} \) and this is clearly ruled out.

Case 2. Suppose \( X^n = Y^m \). Then \( X^nZ^m = 1 \). Thus \( X \) is invertible.

In this case, every element \( s \) of \( S^* \) is a power of \( X \). If not, there exists \( n \geq 1 \) so that \( X^{n+1} < s < X^n \), then \( X < sX^{-n} < 1 \) where \( X^{-1} \) is the inverse of \( X \). This is a contradiction, as \( X \) is the second largest element of \( S^* \). Thus in either case \( S^* = \{ X^n : n \geq 0 \} \) and \( S \) is a group. \( S \) is a discretely ordered group, that is \( S^* \) has a second largest element. Hence \( S \) is order isomorphic to the ordered group of integers.

Proposition (6.3) enables us to obtain a characterization of discrete valuation rings of rank 1.

(6.4) Theorem. Let \((A, S, \sigma)\) be a Hahn valued domain, \((A^*, S^*, \sigma)\) its Hahn valuation ring and \(P\) the Hahn valuation ideal. Assume \(A\) is a Hausdorff topological ring in the Hahn topology and that \(A\) contains \(A^*\) properly. Then \(A\) is a discrete valued field of rank 1 and \(\sigma\) a field valuation if and only if the following conditions are satisfied:

(i) \(A^*\) is a local ring with \(P\) as its maximal ideal.

(ii) \(S^*\) is anti-well ordered and \(\sigma\) of type \(\omega\).

Proof. If \(A\) is a discrete valued field, the conditions hold. On the other hand, since \(A\) is a topological ring, for every \(t \in S\) with \(t > 1\), there exists \(s \in S\) with \(st \leq 1\) (Corollary (4.3)). Thus by Proposition (6.3), \(S\) is an ordered group, order isomorphic to the ordered group of integers. Thus it remains only to prove that \(A\) is a field. If \(a \in A\) and \(a \neq 0\), then \(\sigma(a) = \bar{a}\) is invertible in \(S\) and so there exists \(b \in A\) with \(\bar{a}b = 1\). Thus \(ab \in A^*\) and \(ab \not\in P\), the unique maximal ideal of \(A^*\). Thus \(ab\) is invertible in \(A^*\). So \(a\) is invertible.

(6.5) Corollary. Let \(S\) be a commutative fully ordered group and \(A, A^*, P\) as above. Then \(A\) is a field (and a valued field) if and only if \(A^*\) is a local ring with \(P\) as its maximal ideal.

(6.6) Example. The ring \(Z\langle p \rangle[[X]]\) over the localization of \(Z\) with respect to the prime ideal \((p)\), when endowed with the least degree Hahn valuation shows that \(A^*\) even though local may not have \(P\) as its maximal ideal. We now present the main investigation of this paper.
(6.7) Theorem. Let $S$ be a commutative Hahn semigroup, $(A, S, \sigma)$ a Hahn valued domain and assume that the corresponding Hahn valuation ring $A^*$ is properly contained in $A$. Further let $A$ be a topological ring in the Hahn topology and $A^*$ compact. Then $A$ is a complete discrete valued field of rank 1, $\sigma$ a field valuation and $A^*$ its (discrete) valuation ring. Moreover, the residue class field of $(A, \sigma)$ is finite.

Proof. By Proposition (5.3), $A^*$ is a local ring and $P$ its maximal ideal. Theorem (5.7) assures of condition (ii) of (6.4). Thus (6.4) gives us the result. The last result follows from (6.1). On the other extreme, we have the following result.

(6.8) Theorem. Let $(A, S, \sigma)$ admit non-trivial zero divisors and let $A^*$ be Hausdorff compact. Then $A = A^*$ and $P = N$, that is every element of $P$ is nilpotent.

Proof. Observe that $A^*/N$ has the discrete quotient topology and that it is compact. Hence, $A^*/N$ is a finite field and $\Sigma^* = S^* \setminus \sigma(N)$ is a finite ordered group. Hence $\Sigma^* = (1)$ and $P = N$. $S^*$ has a second largest element, say $s$. If $t > 1$ in $S$ then $ts \ge s$ and $ts \in \sigma(N)$. Thus $ts = s$ and so either $s = 0$ or $t = 1$ and both are not the case. Thus $S = S^*$ and so $A = A^*$.

(6.9) Corollary. Let $(A, S, \sigma)$ be a Hahn valued ring and let $S^*$ be anti-well ordered and of type $\omega$. If $A$ admits non-trivial zero divisors then $A = A^*$.

Proof. Actually it is enough to show that $S = S^*$. Let $S^*$ be 1 $>$ $X_1$ $>$ $X_2$ $>$ $X_3$ $>$ $\ldots$. Let $K$ be a finite field. Consider the ring $B = K[[S^*]]$ of all formal expressions $k = k_0 + \sum_{i=1}^{\infty} k_i X_i$ ($k_0, k_i \in K$). Addition is defined as usual and multiplication by $kk' = k_0 k_0'$. There is a natural Hahn valuation of $B$ and $B$ is compact in the Hahn topology. So by (6.8) $X_1$ is nilpotent in $S^*$ and there exists a smallest integer $n \ge 2$ with $X_1^n = 0$. On the other hand if $Z > 1$ in $S$ then $ZX_1 \in S^*$; also $0 = ZX_1^n = (ZX_1) X_1^{n-1}$ and $X_1^{n-1} \neq 0$. So $ZX_1$ is a zero divisor and hence must belong to $S^*$. In that case $ZX_1 = 1$, but 1 is not a zero divisor. Thus $S = S^*$.

References