PLURI-ADJOINTS AND PRESERVATION OF FINITE LIMITS

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Communicated by J.D. Stasheff
Received 4 April 1988
Revised 6 February 1989

If \( G \) is a functor between two categories \( \mathcal{A} \) and \( \mathcal{B} \), Freyd's Adjoint Functor Theorem provides, under suitable conditions, the connection between the existence of a (left) adjoint \( F \) for \( G \) and the preservation by \( G \) of all small limits (that are assumed to exist in \( \mathcal{A} \)). The present paper deals with the situation when the left adjoint for \( G \) fails to exist, yet some conditions are present that come close to its existence. Thus, for each object \( X \) of \( \mathcal{A} \), a set \( \{ F'(X) \mid r \in T(X) \} \) of objects, with certain properties, is given rather than a single object \( F(X) \). These data and properties define a pluri-adjoint for \( G \). The existence of a pluri-adjoint for \( G \) is shown to be equivalent to the fact that \( G \) preserves finite limits rather than arbitrary small limits. Several examples are provided. In particular, it is shown that the distributive property in a lattice is equivalent to the existence of some pluri-adjoint.

0. Introduction

In what follows, we present a generalization of Freyd's Adjoint Functor Theorem \cite{1;7;19;37,p.117}. According to Mac Lane \cite[p.V]{37}, the slogan is: "Adjoint functors arise everywhere". They occur naturally in powerful universal constructions, such as free objects, tensor products, etc. One of the most important results concerning adjoint functors is Freyd's Adjoint Functor Theorem, which provides, under suitable conditions, a criterion for a functor \( G \) between two categories \( \mathcal{A} \) and \( \mathcal{B} \) to have a (left) adjoint. Namely, Freyd's Theorem states (cf. \cite[§V.6,p.117]{37}) that if \( G : \mathcal{A} \rightarrow \mathcal{B} \) is a functor such that \( \mathcal{A} \) is small-complete and has small hom-sets, then \( G \) has a left adjoint if and only if \( G \) preserves all small limits and satisfies the Solution Set Condition. Now, the Solution Set Condition is intended to keep under control the (possible) big size of \( \mathcal{A} \) and \( \mathcal{B} \). If \( \mathcal{A} \) and \( \mathcal{B} \) are small categories, the Solution Set Condition is automatically satisfied. Throughout this paper, we make the restriction that the categories at issue be small. This is a reasonable restriction, and the loss of generality is counterbalanced by a simplified presentation. Presum-
ably, one can handle categories of arbitrary size, as in the adjoint situation, by working with an analogue of the Solution Set Condition.

There are many important situations when a functor $G : \mathcal{A} \to \mathcal{B}$ fails to have an adjoint. Among such situations we consider those in which the categories $\mathcal{A}$ and $\mathcal{B}$ have finite limits and the functor $G$ preserves them. They are characterized by the existence of a *pluri-adjoint*, a concept we introduce, and which generalizes that of an adjoint (functor). The precise result is provided by Theorem 2.1.

It should be mentioned here that Diers [4, 5] considered a similar situation and solved the corresponding problem by introducing the concept of a (left) multi-adjoint. This is, however, a concept different from our pluri-adjoint and covers a different area of examples and applications: Indeed, in Diers’s approach, if $U : \mathcal{A} \to \mathcal{B}$ is a functor, for each $B \in \text{Ob} \mathcal{B}$, there is a set $I$ and for each $i \in I$ a morphism $g_i : B \to U A_i$ such that for each morphism $g : B \to U A$, $A \in \text{Ob} \mathcal{A}$, there is a unique $i \in I$ and a unique morphism $f : A_i \to A$ with the property that $U f \circ g_i = g$.

And, then, Diers’s analogue of Freyd’s Adjoint Functor Theorem deals with connected limits.

In Section 1, we show that the pluri-adjoint arises in a multitude of cases. Let us begin with the following example, which, when fully developed in Section 1, will illustrate the concept of pluri-adjoint.

Let $\mathcal{A} = \text{Fingr}$ be the category of finite groups and group homomorphisms, and let $\mathcal{B} = \text{Finset}$ be the category of finite sets and their mappings. By suitable restrictive conditions on the size of classes of isomorphic objects, we may assume that these two categories are small. Let $G : \text{Fingr} \to \text{Finset}$ be the forgetful functor. Let us show that $G$ cannot have a left adjoint. (Note, in the first place, that Freyd’s Theorem is not applicable to this case, as arbitrary limits do not exist in $\text{Fingr}$.) Assume, on the contrary, that $G$ does have a (left) adjoint $F$. This would mean that for every nonempty finite set $X$ there would exist a finite group $F(X)$ and a mapping $k : X \to F(X) = GF(X)$ solving the universal arrow problem for the functor $G$; i.e., given any mapping $f : X \to A = GA$, with $A$ a finite group, there would exist a unique group homomorphism $g : F(X) \to A$ making the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{k} & F(X) \\
& \downarrow{f} & \downarrow{g} \\
& & A.
\end{array}
$$

Such a group $F(X)$ cannot exist, as the orders of the different $A$’s are not bounded. If $F(X)$ were to have order $n$, say, we could take $A$ to be a cyclic group of order $n + 1$ with generator $C$ and map all elements of $X$ to $c$. For such a mapping $f$, there could not exist any $g$ making the above diagram commutative.

However, the situation requires a concept that would lead to an analogue of Freyd’s Adjoint Functor Theorem. Indeed, both categories at issue have all finite limits and the forgetful functor $G$ indeed preserves them. The substitute for the
adjoint of $G$ in this case is provided by the pluri-adjoint. The single group $F(X)$ is replaced by a family $(F^\tau(X))_{\tau \in T(X)}$ of finite groups (which is further expanded to a category $\mathcal{F}(X)$), and this family has the properties of inductive injectivity, pseudo-filteredness, and directedness. It fits exactly the given situation. The details are given in Section 1; in Example 1.5, we return to this situation.

Section 2 is devoted to the main theorem (Theorem 2.1).

The pluri-adjoint is present in an especially elegant form in lattice theory. We deal with this setting in Section 3. Let $L$ be a lattice with 0 and 1. Then, for every $a \in L$, the join functor $a \vee - : L \to L$ has a pluri-adjoint if and only if the lattice $L$ is distributive (Corollary 3.4). Thus, distributivity in a lattice is a pluri-adjoint property.

This is analogous to yet another result of Freyd: In a complete lattice, complete distributivity is an adjoint property. In Section 3 we also present a simple example of a functor that does not even have a pluri-adjoint.

As a direction for future investigation, one might try to determine if there is any connection between the concept of pluri-adjoint and a number of concepts in the literature, such as, for example, the different notions of completions of categories [3, 14, 27, 29, 44], profinite categories [24, Chapter VI], dense and codense functors [37, p. 242; 43], etc. Also, the relationship, if any, with the study of various types of semantics [15, 31, 35] and continuous categories [6, 11, 25, 28] could be studied.

1. Definitions and examples

For a category $\mathcal{C}$, we shall denote by $\text{Ob} \mathcal{C}$ the class of objects of $\mathcal{C}$. If $C, D \in \text{Ob} \mathcal{C}$, we shall denote by $\mathcal{C}(C, D)$ the class of morphisms $C \to D$.

We recall that a category is said to be finitely complete if it has all finite limits; and, dually, finitely cocomplete, if it has all finite colimits.

Let $\mathcal{A}$ and $\mathcal{B}$ be small finitely complete categories, and let $G : \mathcal{A} \to \mathcal{B}$ be a functor.

Let us assume that, for every $X \in \text{Ob} \mathcal{B}$, there is given a nonempty set $T(X)$ and, for each $\tau \in T(X)$, an object $F^\tau(X)$ of $\mathcal{A}$ together with a morphism $h_\tau : X \to GF^\tau(X)$ of $\mathcal{A}$. Then $T(X)$ can be expanded into a small category $\mathcal{F}(X)$ having $T(X)$ as its set of objects by taking as morphisms $\sigma \to \tau$ in $\mathcal{F}(X)$, $\sigma, \tau \in T(X)$, those morphisms $r : F^\sigma(X) \to F^\tau(X)$ of $\mathcal{A}$ for which the following diagram commutes:

```
\begin{array}{ccc}
F^\sigma(X) & \to & \to \to \\
\downarrow & \downarrow & \downarrow \\
F^\tau(X) & \to & \to \\
\end{array}
```

Composition and identities are as in $\mathcal{A}$. There obviously is a canonical functor
\( \mathcal{F}(X) \to X \downarrow G \), where \( X \downarrow G \) is the comma category of objects \( G \)-under \( X \) (cf. [37, §11.6, pp. 46–47]), assigning the pair \((F^\tau(X), h_\tau)\) to each object \( \tau \) of \( \mathcal{F}(X) \) and the morphism \( r \) of \( X \downarrow G \) to \( r \) from \( \mathcal{F}(X) \). There also is an obvious faithful functor \( \mathcal{F}(X) \to \mathcal{A} \), given by \( \tau \mapsto F^\tau(X), r \mapsto r \).

Now, we can proceed to the definition of our main concept.

**Definition 1.1.** A (left) pluri-adjoint for \( G \) consists of the following data: For each \( X \in \text{Ob } \mathcal{A} \), there is given a nonempty set \( T(X) \); for each \( \tau \in T(X) \), there is given an object \( P(X) \) of \( \mathcal{A} \) together with a morphism \( h_\tau : X \to GF^\tau(X) \). Then, according to the above construction, \( T(X) \) can be expanded into a small category, \( \mathcal{A}(X) \). These data are to satisfy the following conditions:

(i) **Global covering property.** For each \( g : X \to GA \), there exist at least one \( \tau \in T(X) \) and at least one \( f : F^\tau(X) \to A \) such that \( g \) can be factored as \( g = Gf \circ h_\tau \), \( X \downarrow g \).

\[
\begin{array}{ccc}
X & \xrightarrow{h_\tau} & GF^\tau(X) \\
\downarrow{g} & & \downarrow{Gf} \\
GA. & & \\
\end{array}
\]

(ii) **Inductive injectivity.** Given \( X \in \text{Ob } \mathcal{A} \), \( A \in \text{Ob } \mathcal{A} \), and \( \tau \in T(X) \), if \( f_1, f_2 \in \mathcal{A}(F^\tau(X), A) \) are such that \( Gf_1 \circ h_\tau = Gf_2 \circ h_\tau \), then there are \( \sigma \in T(X) \) and a morphism \( r : \sigma \to \tau \) in \( \mathcal{F}(X) \) (that is, \( r : F^\sigma(X) \to F^\tau(X) \) with \( Gr \circ h_\sigma = h_\tau \)) such that \( f_1 \circ r = f_2 \circ r \).

\[
\begin{array}{ccc}
X & \xrightarrow{h_\sigma} & GF^\sigma(X) \\
\downarrow{h_\tau} & & \downarrow{Gr} \\
GF^\tau(X) & \xrightarrow{Gf_1} & GA. \\
\end{array}
\]

(iii) **Pseudo-filteredness.** Given \( X \in \text{Ob } \mathcal{A} \), each diagram

\[
\begin{array}{ccc}
& & \nu \\
& v & \leftarrow u \\
\eta & \downarrow {u'} & \tau \\
& \downarrow {v'} & & \downarrow v \\
& \varrho & \xrightarrow{v} \nu. \\
\end{array}
\]

in \( \mathcal{F}(X) \) can be filled in as a commutative diagram in \( \mathcal{F}(X) \),

\[
\begin{array}{ccc}
\eta & \xrightarrow{u'} & \tau \\
\downarrow {v'} & & \downarrow u \\
\varrho & \xrightarrow{v} \nu. \\
\end{array}
\]

(iv) **Directedness.** Given \( X \in \text{Ob } \mathcal{A} \), for all \( \tau, \varrho \in T(X) \), there is at least one
Phi-adjoints and preservation of finite limits

For small finitely cocomplete categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $G: \mathcal{A} \to \mathcal{B}$, a right pluri-adjoint for $G$ consists of the following data: For each $X \in \text{Ob } \mathcal{A}$, there is given a nonempty set $T(X)$; for each $T \in T(X)$, there is given an object $F^*(X)$ of $\mathcal{A}$ together with a morphism $k_T : G\cdot F^*(X) \to X$. If $T(X)$ is expanded into a small category $\mathcal{F}(X)$ according to a procedure dual to the one described above, these data are to satisfy conditions (i')-(iv') dual to (i)-(iv).

Thus, in case $G$ has a left adjoint, for each object $X$ of $\mathcal{A}$, there is an object $F(X)$ of $\mathcal{A}$ and a morphism $\eta_X : X \to GF(X)$ (the unit of the adjunction) that is universal for the morphisms $X \to GA$; if $G$ has a (left) pluri-adjoint, there is a category $\mathcal{F}(X)$ and, for each $T \in \text{Ob } \mathcal{F}(X)$, an object $F^*(X)$ of $\mathcal{A}$ and a morphism $h_T : X \to GF^*(X)$, and the $h_T$'s globally solve the universal problem for the morphisms $X \to GA$.

Remark 1.2. If $G: \mathcal{A} \to \mathcal{B}$ has a (left) pluri-adjoint, then, for each $A \in \text{Ob } \mathcal{A}$, $X \in \text{Ob } \mathcal{A}$, and $T \in T(X)$, one can define a mapping

$$\Phi(X, \tau, A) : \mathcal{A}(F^*(X), A) \to \mathcal{B}(X, G(A))$$

by setting $\Phi(X, \tau, A)(f) = Gf \circ h_T$ for $f \in \mathcal{A}(F^*(X), A)$. The mappings $\Phi(X, \tau, A)$ are natural in $A$, and one can state the definition of the pluri-adjoint in terms of these mappings.

Now, we prove some direct consequences of the definition. The next lemma shows us that two morphisms from $X$ to $G(A)$ can be lifted simultaneously.

Lemma 1.3. Let the functor $G: \mathcal{A} \to \mathcal{B}$ have a pluri-adjoint as in Definition 1.1. Given $X \in \text{Ob } \mathcal{A}$, $A \in \text{Ob } \mathcal{A}$, and a pair of morphisms $g_1, g_2 : X \to G(A)$, there exist $\eta \in T(X)$ and morphisms $f_1, f_2 : F^*(X) \to A$ such that $Gf_i \circ h_\eta = g_i$, $i = 1, 2$.

Proof. Using the global covering property (Definition 1.1(i)), one can find morphisms $k_i : F^*(X) \to A$, with $\tau_i \in T(X)$, such that

$$g_i = G(k_i) \circ h_\eta, \quad i = 1, 2. \tag{1}$$

By directedness (Definition 1.1(iv)), there are $\eta \in T(X)$ and morphisms

$$\tau_1 \leftarrow \eta \leftarrow \tau_2$$

in $\mathcal{F}(X)$. That is,
Now, for $i=1,2$, we have
\[
g_i = G(k_i) \circ h_i \quad \text{ (by (1))}
\]
\[
= G(k_i) \circ G(u_i) \circ h_i \quad \text{ (since $u_i$ is a morphism of the category $\mathcal{F}(X)$)}.
\]

Taking $f_i = k_i \circ u_i$, $i=1,2$, the conclusion follows. □

**Lemma 1.4.** Assume that the functor $G: \mathcal{A} \rightarrow \mathcal{B}$ has a pluri-adjoint as in Definition 1.1. Let $X \in \text{Ob} \; \mathcal{A}$, $A_i \in \text{Ob} \; \mathcal{A}$, $i=1,2$, $\sigma \in T(X)$, and let $f_i, f'_i: F^\sigma(X) \rightarrow A_i$ be morphisms such that $Gf_i \circ h_i = Gf'_i \circ h_i$, $i=1,2$. Then there exist $\eta \in T(X)$ and a morphism $s: \eta \rightarrow \sigma$ in $\mathcal{F}(X)$ such that, in $\mathcal{A}$, $f_i \circ s = f'_i \circ s$, $i=1,2$.

**Proof.** Using the inductive injectivity (Definition 1.1(ii)), one can find objects $v_1, v_2$ of $\mathcal{F}(X)$ and morphisms $l_i: v_i \rightarrow \sigma$ in $\mathcal{F}(X)$ such that, in $\mathcal{A}$,
\[
f_i \circ l_1 = f'_i \circ l_1, \quad i=1,2.
\]

Applying the pseudo-filteredness property (Definition 1.1(iii)), there are $\eta \in T(X)$ and morphisms
\[
v_1 \leftarrow r_1 \quad \eta \quad r_2 \rightarrow v_2
\]
in $\mathcal{F}(X)$ such that
\[
\eta \quad r_1 \quad v_1 \quad r_2 \quad s \quad l_1 \quad v_2 \quad l_2 \quad \sigma
\]
commutes \((l_1 \circ r_1 = l_2 \circ r_2 = s, \text{ say})\). The following diagram in \(\mathcal{A}\) will illustrate the situation:

![Diagram](https://via.placeholder.com/150)

We have for \(i = 1, 2\),

\[
    f_i \circ s = f_i \circ l_i \circ r_i \\
    = f'_i \circ l_i \circ r_i \quad \text{(by (2))} \\
    = f'_i \circ s.
\]

If the functor \(G: \mathcal{A} \to \mathcal{B}\), with \(\mathcal{A}, \mathcal{B}\) small finitely cocomplete categories, has a right pluri-adjoint, then the duals \(1.3', 1.4'\) of Lemmas 1.3, 1.4, respectively, are valid.

Now, we turn to the examples. Our first example is the one mentioned in the introduction.

**Example 1.5.** Let \(G: \text{Fingr} \to \text{Finset}\) be the forgetful functor from the category of finite groups to the category of finite sets (with restrictions on the size of isomorphism classes, so as to make them small categories). We showed in the introduction that \(G\) cannot have a left adjoint. We will show, however, that \(G\) does have a pluri-adjoint. For a finite set \(X\), define \(T(X)\) as the set of all finite quotient groups of the free group \(F(X)\) on \(X\). If \(r \in T(X)\), take \(F'(X)\) to be \(r\). The mapping \(h_i: X \to GP(X)\) is defined as the composite \(X \xrightarrow{i} F(X) \xrightarrow{p'} F'(X)\), where \(i\) is the inclusion mapping and \(p'(X) = p^T\) is the canonical surjection (note that, if regarded as sets, \(GA = A\) for every finite group \(A\)).

We must show that the properties stated in Definition 1.1 hold. To check the global covering property, let a mapping \(g: X \to GA\) be given, where \(X\) is a finite set and \(A\) a finite group. Then \(g\) can be uniquely extended to a group homomorphism \(g': F(X) \to A\) such that \(g' \circ i = g\). Let \(F^T(X) = F(X)/(\text{Ker } g')\). Then \(g'\) factors through \(F^T(X)\), yielding an injective group homomorphism \(f: F^T(X) \to A\) such that \(g' = f \circ p^T\). The diagram is

\[
    \begin{array}{ccc}
    X & \xrightarrow{i} & F(X) \\
    g \downarrow & & \downarrow p^T(X) \\
    A & \xleftarrow{f} & F^T(X) = F(X)/(\text{Ker } g').
    \end{array}
\]
Thus, $g$ factors as $g = f \circ p^*(X) \circ i = f \circ h_\tau = Gf \circ h_\tau$.

For the inductive injectivity, assume that $f_1, f_2 : F^\tau(X) \to A$ are group homomorphisms such that $Gf_1 \circ h_\tau = Gf_2 \circ h_\tau$. But then $f_1 \circ p^*(X)$ and $f_2 \circ p^*(X)$ are homomorphisms coinciding on the generating set $X$, and, hence, they are equal. Consequently, $f_1 = f_2$ since $p^*$ is a surjection.

For the pseudo-filteredness, consider the diagram in $\mathcal{T}(X)$,

$$
\begin{array}{c}
\mathcal{T} \\
\downarrow^u \\
\mathcal{F}'(X) \\
\downarrow^v \\
\mathcal{F}^\tau(X)
\end{array}
$$

This translates into the diagram in $\mathcal{F}'(X)$,

$$
\begin{array}{c}
\mathcal{F}'(X) \\
\downarrow^u \\
\mathcal{F}^\tau(X) \\
\downarrow^v \\
\mathcal{F}^\sigma(X)
\end{array}
$$

with the property that $Gu \circ h_\tau = h_\nu$ and $Gu \circ h_\tau = h_\nu$. By the same kind of argument as above, it follows that

$$v \circ p^\sigma(X) = p^\tau(X) = u \circ p^\tau(X). \quad (3)$$

Consequently, $\text{Ker } p^\tau \subseteq \text{Ker } p^\nu$ and $\text{Ker } p^\rho \subseteq \text{Ker } p^\nu$. Since $\text{Ker } p^\tau$ and $\text{Ker } p^\rho$ have finite index in $F(X)$, so does their intersection $N$ (cf. [20, Exercise 12, p. 47]). Let $F^\eta(X) = F(X)/N$. Then $F^\eta(X)$ is a member of $T(X)$. Since $N \subseteq \text{Ker } p^\tau$ and $N \subseteq \text{Ker } p^\rho$, there are canonical surjections $u' : F^\eta(X) \to F^\tau(X)$ and $v' : F^\eta(X) \to F^\rho(X)$ such that $u' \circ p^\sigma(X) = p^\tau(X)$ and $v' \circ p^\sigma(X) = p^\rho(X)$. Consequently, $Gu' \circ h_\eta = h_\tau$ and $Gu' \circ h_\eta = h_\rho$. This means that $u' : \eta \to \tau$ and $v' : \eta \to \varrho$ are morphisms of the category $\mathcal{T}(X)$. On the other hand, by virtue of (3), $u$ and $v$ are the canonical surjections $F(X)/(\text{Ker } p^\tau(X)) \to F(X)/(\text{Ker } p^\nu(X))$ and $F(X)/(\text{Ker } p^\rho(X)) \to F(X)/(\text{Ker } p^\nu(X))$, respectively. Hence the diagram

$$
\begin{array}{c}
\mathcal{T} \\
\downarrow^u \\
\mathcal{F}'(X) \\
\downarrow^v \\
\mathcal{F}^\tau(X)
\end{array}
$$

commutes in $\mathcal{T}(X)$ since the corresponding diagram commutes in $\mathcal{F}'(X)$.

Finally, directedness follows directly from pseudo-filteredness. For, if $\tau, \varrho \in T(X)$ are given, one can take for $v$ (actually, $F^\nu(X)$) the identity group $E$ (clearly, a finite quotient group of $F(X)$), with $u : F^\tau(X) \to F^\nu(X)$, $v : F^\rho(X) \to F^\nu(X)$, the unique ('zero') homomorphisms. Since $GF^\nu(X)$ consists of a single element, $Gu \circ h_\tau = h_\nu$ and $Gu \circ h_\rho = h_\nu$, so that $u : \tau \to v$ and $v : \varrho \to v$ are morphisms of $\mathcal{T}(X)$.

Thus, all the properties of the pluri-adjoint have been verified.

The next example is from the domain of abelian groups.

**Example 1.6.** Let $\mathcal{A}$ be a small full subcategory of $\textbf{Ab}$ whose objects are torsion abelian groups and which is closed under formation of subgroups, quotient groups,
and finite direct products (hence also finite direct sums), e.g., the full subcategory of \( \text{Ab} \) of all at most countable torsion abelian groups, with a suitable condition on the sizes of isomorphism classes, so as to obtain a small category. Let \( \mathcal{A} \) be the full subcategory of \( \mathcal{A} \) whose objects are those torsion abelian groups from \( \mathcal{A} \) that have finitely many primary components. Then \( \mathcal{A} \) and \( \mathcal{A} \) are finitely complete.

Let \( G: \mathcal{A} \to \mathcal{A} \) be the imbedding functor. An argument analogous to the one in the introduction can be used to prove that \( G \) does not have a left adjoint. But, again, \( G \) has a pluri-adjoint. If \( X \in \text{Ob} \mathcal{A} \), then \( X \) is a torsion abelian group. Let \( T(X) \) be the (nonempty) set of all subgroups of \( X \) that are sums of finitely many primary components of \( X \). For \( \tau \in T(X) \), define \( F^\tau(X) = \tau \). Since \( F^\tau(X) \) is a direct summand of \( X \), there is a canonical surjection \( p^\tau(X): X \to F^\tau(X) = GF^\tau(X) \). If \( p^\tau(X) \) is taken for the \( h_\tau \) from the definition of the pluri-adjoint, it is easy to verify that these data define indeed a pluri-adjoint for \( G \).

It is perhaps interesting to note that the functor \( G \) of Example 1.6 has also a right pluri-adjoint.

**Example 1.6'.** Let \( \mathcal{A} \) and \( \mathcal{A} \) be as in Example 1.6. Take, for an object \( X \) of \( \mathcal{A} \), the same family \( (F^\tau(X))_{\tau \in T(X)} \) as in Example 1.6. Define the mapping \( k_\tau: GF^\tau(X) \to X \), \( \tau \in T(X) \), as being the inclusion mapping \( F^\tau(X) \to X \).

There are other situations when a functor has a right pluri-adjoint but not a right adjoint.

**Example 1.7.** This example is also from the domain of abelian groups. Let \( \mathcal{A} \) be a small full subcategory of \( \text{Ab} \) which is closed under formation of subgroups, quotient groups, and finite direct products, e.g., the full subcategory of \( \text{Ab} \) of all at most countable abelian groups, with the usual restriction on the sizes of isomorphism classes. Let \( \mathcal{A} \) be the full subcategory of \( \mathcal{A} \) whose objects are the finitely generated abelian groups from \( \mathcal{A} \). Let \( G: \mathcal{A} \to \mathcal{A} \) be the imbedding functor. If \( X \in \text{Ob} \mathcal{A} \), let \( (F^\tau(X))_{\tau \in T(X)} \) be the nonempty family of all finitely generated subgroups of \( X \). For \( \tau \in T(X) \), define the mapping \( k_\tau: GF^\tau(X) \to X \) as the inclusion mapping \( F^\tau(X) \to X \). It is easy to verify that these data define a right pluri-adjoint for \( G \).

An analogous situation occurs when we take for \( \mathcal{A} \) a small category, having kernels, cokernels, and finite products, of left \( R \)-modules over a ring \( R \) with identity and for \( \mathcal{A} \) the full subcategory of finitely generated modules from \( \mathcal{A} \); The embedding functor \( G \) has a right pluri-adjoint. (**Question:** When does \( G \) have a right adjoint?)

**Remark 1.8.** An alternative way of defining the concept of pluri-adjoint is the following:
The functor \( G : \mathcal{A} \to \mathcal{B} \) has a left pluri-adjoint if for each \( X \in \text{Ob} \mathcal{B} \) there is a subcategory \( \mathcal{F}(X) \) of \( \mathcal{A} \) satisfying the following properties:

(i) For each \( a \in \text{Ob}(\mathcal{F}(X)) \) there is a specified map \( h_a : X \to Ga \).

(ii) \( r : a \to b \) is a map of \( \mathcal{F}(X) \) if and only if \( Gr \circ h_a = h_b \).

(iii) Pseudo-filteredness. Every diagram \( a \to b \to c \) in \( \mathcal{F}(X) \) can be filled in as

\[
\begin{array}{ccc}
d & \rightarrow & c \\
\downarrow & & \downarrow \\
a & \rightarrow & b.
\end{array}
\]

(iv) Directedness. For all \( a, b \in \text{Ob}(\mathcal{F}(X)) \) there is a diagram \( a \leftarrow c \to b \) in \( \mathcal{F}(X) \).

(v) Global covering property. For each \( g : X \to GA \) there exists at least one factorization of \( g \) as \( Gf \circ h_a \) for some \( a \in \text{Ob}(\mathcal{F}(X)) \) and some \( f : a \to A \).

(vi) Inductive injectivity. For all \( X \) and for all \( f_1, f_2 : a \to A \), if \( Gf_1 \circ h_a = Gf_2 \circ h_a \), then there exists a map \( r : b \to a \) in \( \mathcal{F}(X) \) such that \( f_1 \circ r = f_2 \circ r \). Diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h_a} & Ga \\
\downarrow & & \downarrow \\
Gh_a & \xrightarrow{Gr} & GA.
\end{array}
\]

2. The main theorem

The following theorem is the pluri-adjoint analogue of Freyd's Adjoint Functor Theorem (under the assumption that we are dealing just with small categories).

**Theorem 2.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be small (nonempty) finitely complete categories and \( G : \mathcal{A} \to \mathcal{B} \) a functor. Then \( G \) preserves finite limits if and only if \( G \) has a (left) pluri-adjoint.

Theorem 2.1 will follow as a consequence of Lemmas 2.2-2.5.

Throughout this section, \( \mathcal{A} \) and \( \mathcal{B} \) will be small categories as in the statement of Theorem 2.1, and \( G : \mathcal{A} \to \mathcal{B} \) will be a functor.

**Lemma 2.2.** If the functor \( G \) has a pluri-adjoint, then \( G \) preserves finite products.

**Proof.** Assume \( G \) has a (left) pluri-adjoint as in Definition 1.1. We only need to prove that \( G \) preserves the terminal object and products of two objects.

Let \( 1_\mathcal{A} \) be the terminal object of \( \mathcal{A} \). We shall show that \( G(1_\mathcal{A}) \) is a terminal object of \( \mathcal{B} \). Let \( X \in \text{Ob} \mathcal{B} \). There exists some \( \tau \in T(X) \) since \( T(X) \) is nonempty. There is at
least one morphism \( X \to G(1, \mathcal{A}) \), namely, the composite \( X \xrightarrow{h} GF^T(X) \to G(1, \mathcal{A}) \), where the second arrow is the image under \( G \) of the unique morphism \( F^T(X) \to 1, \mathcal{A} \).

If \( g_1, g_2 \) are two morphisms \( X \to G(1, \mathcal{A}) \), then, by Lemma 1.3, there exist \( \eta \in T(X) \) and morphisms \( f_1, f_2 : F^\eta(X) \to 1, \mathcal{A} \) such that \( Gf_i \circ h = g_i \), \( i = 1, 2 \). Since \( 1, \mathcal{A} \) is a terminal object, we must have \( f_1 = f_2 \), whence \( g_1 = g_2 \). Thus, there is but one morphism \( X \to G(1, \mathcal{A}) \).

Now, let \( A_1, A_2 \in \text{Ob} \mathcal{A} \), and let

\[
\begin{array}{c}
\text{A}_2 \\
\downarrow p_2 \\
\text{A} \\
\downarrow p_1 \\
\text{A}_1
\end{array}
\]

be a product of \( A_1 \) and \( A_2 \). Let \( X \in \text{Ob} \mathcal{A} \), and let the diagram

\[
\begin{array}{ccc}
\text{X} & \xrightarrow{g} & \text{G(A)} \\
\downarrow g_1 & & \downarrow G(p_1) \\
\text{G(A}_1) \\
\downarrow G(p_2) & & \\
\text{G(A}_2)
\end{array}
\]

be given. We must find the unique arrow \( g : X \to G(A) \) making diagram (4) commutative. By the global covering property (Definition 1.1(i)), there are elements \( \tau_1, \tau_2 \in T(X) \) and morphisms \( f_i : F^\tau_i(X) \to A_i \) such that

\[
Gf_i \circ h_i = g_i, \quad i = 1, 2.
\]

By directedness (Definition 1.1(iv)), there are \( \sigma \in T(X) \) and morphisms

\[
\begin{array}{ccc}
\tau_1 & \xrightarrow{k_1} & \sigma \\
\downarrow k_2 & & \downarrow k_3 \\
\sigma
\end{array}
\]

in the category \( \mathcal{A}(X) \). (In \( \mathcal{A} \), these translate into \( k_i : F^\sigma(X) \to F^\tau_i(X) \) such that \( Gk_i \circ h_a = h_{\tau_i}, \quad i = 1, 2 \).) Since \( (A_1, p_1, p_2) \) is a product of \( A_1 \) and \( A_2 \), there is a unique arrow \( k = \langle f_1 \circ k_1, f_2 \circ k_2 \rangle \) making the following diagram commute:
Let \( g = gk \circ h \in \mathcal{X}(X, GA) \). We will show that \( g \) makes diagram (4) commute. Indeed, for \( i = 1, 2 \), we have

\[
G(p_i) \circ g = G(p_i) \circ G(k) \circ h = G(f_i \circ k_i) \circ h, \quad \text{(by the commutativity of (6))}
\]

\[
= G(f_i) \circ h_{i_i} \quad \text{(since \( k_i \) is a morphism of \( \mathcal{X}(X) \))}
\]

\[
= g_i \quad \text{(by (5))}.
\]

Now, we must prove that \( g \) is unique. Let \( g, g' \) be morphisms \( X \to G(A) \) such that

\[
G(p_i) \circ g = G(p_i) \circ g' = g_i, \quad i = 1, 2. \quad (7)
\]

By Lemma 1.3, there are morphisms \( f, f' : F^\sigma(X) \to A \) for some \( \sigma \in T(X) \) such that \( Gf \circ h_\sigma = g, \ Gf' \circ h_\sigma = g' \). Then, by (7), \( G(p_i \circ f) \circ h_\eta = g_i = G(p_i \circ f') \circ h_\eta, \ i = 1, 2. \)

By Lemma 1.4, there exist \( \eta \in T(X) \) and a morphism \( s : \eta \to \sigma \) in \( \mathcal{X}(X) \) (that is, \( s : F^\eta(X) \to F^\sigma(X) \) satisfies \( Gs \circ h_\eta = h_\sigma \)) such that \( p_i \circ f \circ s = p_i \circ f' \circ s, \ i = 1, 2. \) We have the following diagram:

\[
\begin{array}{ccc}
F^\eta(X) & \xrightarrow{\ s \ } & F^\sigma(X) & \xrightarrow{\ f \ } & A \\
& ^{p_1} & ^{p_2} & ^{p_1} & ^{p_2} \\
& & & A_1 & \xrightarrow{\ f \ } & A_2.
\end{array}
\]

Since \((A, p_1, p_2)\) is a product of \( A_1 \) and \( A_2 \), we must have \( f \circ s = f' \circ s \). Since \( s \) is a morphism of \( \mathcal{X}(X) \), we have \( g = Gf \circ h_\sigma = Gf' \circ Gs \circ h_\eta \) and, similarly, \( g' = Gf' \circ Gs \circ h_\sigma \). It follows that \( g = g' \). 

**Lemma 2.3.** If the functor \( G : \mathcal{A} \to \mathcal{X} \) has a pluri-adjoint, then \( G \) preserves equalizers of pairs of morphisms.

**Proof.** Assume \( G \) has a pluri-adjoint as in Definition 1.1. Let \( u, v \in \mathcal{A}(A, B) \) be two morphisms, and let \((E, e)\) be an equalizer of \( u \) and \( v \).
We will show that $(G(E), G(e))$ is an equalizer of $G(u)$ and $G(v)$. Clearly, $G(u) \circ G(e) = G(v) \circ G(e)$. Consider a morphism $g : X \to G(A)$ such that
\[ G(u) \circ g = G(v) \circ g. \tag{8} \]

We have the diagram
\[
\begin{array}{ccc}
G(E) & \xrightarrow{G(e)} & G(A) \\
\downarrow{G(u)} & & \downarrow{G(v)} \\
X & \xrightarrow{g} & G(B).
\end{array}
\tag{9}
\]

Our goal is to find a unique morphism $l : X \to G(E)$ making diagram (9) commutative. By the global covering property (Definition 1.1(i)), one can find a morphism $f : F^\tau(X) \to A$ for some $\tau \in T(X)$ such that
\[ g = Gf \circ h_\tau. \tag{10} \]

Then, by (8), we obtain $G(u \circ f) \circ h_\tau = G(u) \circ g = G(v) \circ g = G(v \circ f) \circ h_\tau$. By inductive injectivity (Definition 1.1(ii)), there exist $\sigma \in T(X)$ and a morphism $j : \sigma \to \tau$ in $T(X)$ (that is, $j : F^\sigma(X) \to F^\tau(X)$ from $\mathcal{A}$ satisfies $Gj \circ h_\sigma = h_\tau$) such that $u \circ f \circ j = v \circ f \circ j$. Since $(E,e)$ is an equalizer of $u$ and $v$, it follows that there is a unique morphism $k : F^\sigma(X) \to E$ such that
\[ f \circ j = e \circ k. \tag{11} \]

Define $l = Gk \circ h_\sigma$. We have
\[
G(e) \circ l = G(e) \circ G(k) \circ h_\sigma \\
= G(f) \circ G(j) \circ h_\sigma \quad \text{(by (11))} \\
= G(f) \circ h_\tau \quad \text{(since $j$ is a morphism of $\mathcal{A}$)} \\
= g \quad \text{(by (10)).}
\]

For the uniqueness of $l$, assume that there are two morphisms $l, l' : X \to G(E)$ such that
\[ G(e) \circ l = G(e) \circ l' = g. \tag{12} \]
By Lemma 1.3, there exist $\sigma \in T(X)$ and morphisms $t, t' \in \mathcal{A}(F^\sigma(X), E)$ such that

$$Gt \circ h_\sigma = l \quad \text{and} \quad Gt' \circ h_\sigma = l'.$$

(13)

Using (12) and (13), we get $G(e \circ t) \circ h_\sigma = G(e) \circ l = G(e) \circ l' = G(e \circ t') \circ h_\sigma$. By inductive injectivity (Definition 1.1(ii)), there exist $\eta \in T(X)$ and a morphism $r: \eta \to \sigma$ in $\mathcal{A}(X)$ (that is, $r: F^\eta(X) \to F^\sigma(X)$ in $\mathcal{A}$ with the property that $Gr \circ h_\eta = h_\sigma$) such that $e \circ t \circ r = e \circ t' \circ r$. But $e$ is monic since $(E, e)$ is an equalizer. Thus, $t \circ r = t' \circ r$. On the other hand, using (13) and the fact that $r$ is a morphism of $\mathcal{A}(X)$, we obtain $l = Gt \circ h_\sigma = Gt \circ Gr \circ h_\eta$ and, similarly, $l' = Gt' \circ Gr \circ h_\eta$. It follows that $l = l'$. The proof of uniqueness is complete. $\square$

We need just one more lemma for the proof of the sufficiency statement in Theorem 2.1. Its proof will be omitted, as it is the finite analogue of the corresponding result for arbitrary small limits (for the way the limit of a functor is constructed starting from products and equalizers of pairs of arrows, cf. [37, §V.2, Theorem 1, p. 109; also Theorem 2 and Corollary 1].

Lemma 2.4. Let $\mathcal{C}, \mathcal{D}$ be (nonempty) small finitely complete categories and $G: \mathcal{C} \to \mathcal{D}$ a functor preserving finite products and equalizers of pairs of morphisms. Then $G$ preserves all finite limits. $\square$

Next, we take up the proof of the necessity statement in Theorem 2.1. To this purpose, we recall (cf. [37, §II.6, pp. 46–47]) that if $\mathcal{A}$ and $\mathcal{B}$ are categories (not necessarily small and/or finitely complete) and $G: \mathcal{A} \to \mathcal{B}$ is a functor, the comma category $X \downarrow G$ of objects $G$-under $X$ is defined for every object $X$ of $\mathcal{B}$ as follows: Its objects are pairs $(A, u)$, where $A$ is an object of $\mathcal{A}$ and $u: X \to GA$ is a morphism (in $\mathcal{B}$). If $(A, u), (B, v)$ are two such objects, a morphism $(A, u) \to (B, v)$ in $X \downarrow G$ is a morphism $f: A \to B$ of $\mathcal{A}$ such that $Gf \circ u = v$. The diagram is

\[
\begin{array}{ccc}
    X & & \downarrow Gf \\
    \downarrow v & & \\
    GB & \to & B \\
\end{array}
\]

Composition and identities are as in $\mathcal{A}$. We have the canonical functor

$$Q: X \downarrow G \to \mathcal{A}$$

sending $(A, u)$ to $A$ and $f$ to $f$, for $f: (A, u) \to (B, v)$.

Lemma 2.5. Let $\mathcal{A}, \mathcal{B}$, and $G$ be as in the statement of Theorem 2.1. If the functor $G$ preserves finite limits, then $G$ has a (left) pluri-adjoint.
Proof. Let $X \in \text{Ob } \mathscr{B}$. Since $\mathscr{A}$ and $\mathscr{B}$ are small categories, so is the comma category $X \downarrow G$. It is nonempty. Since if $1_{\mathscr{A}}$ is the terminal object of $\mathscr{A}$, then $G(1_{\mathscr{A}})$ is a terminal object of $\mathscr{B}$, and hence $X \to G(1_{\mathscr{A}})$ is an object of $X \downarrow G$.

To define the pluri-adjoint (in the notation of Definition 1.1), we start by letting $T(X)$ be the nonempty set of all finite subcategories of $X \downarrow G$. Let $\tau \in T(X)$. The following commutative diagram shows a typical morphism $\kappa$ of $\tau$:

$$
\begin{array}{ccc}
G\mathbb{A}_{\alpha} & \xrightarrow{A_{\alpha}} & \mathbb{A}_{\alpha} \\
\downarrow Gk & & \downarrow \kappa \\
G\mathbb{A}_{\beta} & \xrightarrow{A_{\beta}} & \mathbb{A}_{\beta}.
\end{array}
$$

Applying the canonical functor $Q$, one obtains a finite inverse system $Q(\tau)$ in $\mathscr{A}$, whose category of indices is $\tau$. Since $\mathscr{A}$ is finitely complete, $\lim Q(\tau)$ exists. Let us denote it by $L_{\tau}$, with projections $p_{a}: L_{\tau} \to A_{\alpha}$. Since $G: \mathscr{A} \to \mathscr{B}$ preserves all finite limits, it follows that $Q: X \downarrow G \to \mathscr{A}$ creates all finite limits (cf. [37, §V.6, Lemma p. 117]). Thus, there is a unique object $(M_{\tau}, h_{\tau})$ of $X \downarrow G$ and a unique cone $\kappa : (M_{\tau}, h_{\tau}) \to (\tau)$ such that $Q(M_{\tau}, h_{\tau}) = L_{\tau}$ and $Q(\kappa)$ has components $p_{a}$, and, in addition, $\kappa$ is a limiting cone in $X \downarrow G$. But then $M_{\tau} = A_{\tau}$. Thus, $(L_{\tau}, h_{\tau})$ is a limit of $(\tau)$ in $X \downarrow G$ with projections $p_{a}$. We have the following diagram in $X \downarrow G$ with projections $p_{a}$.

We have the following diagram in $X \downarrow G$ (which translates into a commutative diagram in $\mathscr{B}$):

Now, we complete the definition of the pluri-adjoint by setting $F^{\tau}(X) = L_{\tau}$ with the morphism $h_{\tau}: X \to GL_{\tau}$. Then $T(X)$ is expanded into a small category $\mathcal{Y}(X)$ in the manner shown in Section 1. (Note that the morphisms between subcategories $\sigma$ and $\tau$ are not functors.)

We must show that properties (i)-(iv) of Definition 1.1 hold. For the remainder of the proof, $A$ will denote an object of $\mathscr{A}$ and $X$ an object of $\mathscr{B}$.

(i) Global covering property. Let $g: X \to G(A)$ be a morphism. We take for $\tau$ the finite subcategory of $X \downarrow G$ whose only object is the pair $(A, g)$ and whose only morphism is the identity of $(A, g)$. Clearly, the limit $L_{\tau} = F^{\tau}(X)$ is $A$ with projection...
id₄ : F(X) → A. Thus, hᵣ = g, which factors as g = G(id₄) ∘ hᵣ. Consequently, g is lifted to id₄.

(ii) Inductive injectivity. Let τ ∈ T(X), and let f₁, f₂ : F'(X) → A be such that Gf₁ ∘ hᵣ = Gf₂ ∘ hᵣ = l (say). We have the following diagram, in which both triangles commute:

```
  F'(X) ----------> F(X)
   |                 |
   |                 |
  X ---------->    |
   |     |
  l   |
      |    |
      |     |
      |     |
  ↓     |
  G A  ----------> A.
```

Let σ be the subcategory of X↓G whose objects are the two pairs (F'(X), hᵣ) and (A, l) and whose morphisms are, besides identities, f₁ and f₂. Since A is finitely complete, f₁ and f₂ have an equalizer, say, (E, e). Thus, lim Q(σ) = E = Fθ(X), with projections Ge and Gf₁ ∘ Ge (= Gf₂ ∘ Ge). Since the functor Q creates all finite limits, there is a unique morphism h₉ : X → GE (the limit of σ in X↓G) such that, in the following diagram, all triangles commute:

```
  GF'(X) ---------->  
    |                 |
    |                 |
  X ---------->    |
    |     |
  h₉    |     |
    |     |
    |     |
  ↓     |
  GE  ----------> GE
      |     |
      |     |
      |     |
      |     |
  ↓     |
  G A  ----------> A.
```

This means that e is a morphism σ → τ in ℱ(X). And, of course, f₁ ∘ e = f₂ ∘ e.

(iii) Pseudo-filteredness. Let X ∈ Ob ℱ, and let the diagram in ℱ(X),

```
  g  ----------> v  ---------->  
    |     |     |
    |     |     |
  X  ---h₉---> GE  ---h₉--->
    |     |     |
    |     |     |
    |     |     |
    |     |     |
  ↓     |
  G A  ----------> A.
```

be given. This translates into the following commutative diagram in ℱ:

```
  X ----------> GF'(X)
    |     |     |
    |     |     |
  h₉    |     |
    |     |
    |     |
    |     |
    |     |
  ↓     |
  GFθ(X)  ----------> GFv(X).
```

Let η be the finite subcategory of X↓G whose objects are (Fᵢ(X), hᵢ) with i ∈
\{\tau, \rho, \upsilon\} \text{ and whose morphisms are, besides identities, } u \text{ and } \upsilon. \text{ Clearly, the limit }\lim Q(\eta) = F^\upsilon(X) \text{ is the pullback of the diagram}

\[
F^\rho(X) \xrightarrow{\upsilon} F^\upsilon(X) \xleftarrow{\mu} F^\upsilon(X),
\]

with projections, say, \(u'\) and \(v'\). This yields the commutative diagram

\[
\begin{array}{ccc}
F^\rho(X) & \xrightarrow{u'} & F^\upsilon(X) \\
\downarrow & & \downarrow \\
F^\rho(X) & \xrightarrow{\upsilon} & F^\upsilon(X)
\end{array}
\]  
(14)

Again, since the canonical functor \(Q\) creates finite limits, it follows that \(u'\) and \(v'\) are morphisms of the category \(\mathcal{F}(X)\) (that is, in \(\mathcal{A}\), \(G\upsilon' \circ h_{\eta} = h_{\tau}\), \(G\upsilon' \circ h_{\eta} = h_{\rho}\)). And, of course, the diagram in \(g(X)\) obtained from (14) is also commutative.

(iv) **Directedness.** This is a particular case of (iii). Indeed, given \(X \in \text{Ob } \mathcal{A}\) and \(\tau, \rho \in T(X)\), one can take for \(\nu\) the finite subcategory of \(X \downarrow G\) whose only object is \(X \rightharpoonup G(1, \sigma)\), where \(1, \sigma\) is the terminal object of \(\mathcal{A}\), with just the identity morphism. Then \(F^\upsilon(X) = \lim Q(\nu) = 1, \sigma\), and there are unique morphisms \(u : F^\tau(X) \rightarrow F^\upsilon(X)\) and \(\nu : F^\rho(X) \rightarrow F^\upsilon(X)\). Since \(G(1, \sigma)\) is a terminal object of \(\mathcal{A}\) (\(G\) preserves finite limits), \(u\) and \(\upsilon\) are morphisms \(\tau \rightarrow \upsilon\) and \(\rho \rightarrow \upsilon\), respectively, in \(\mathcal{F}(X)\).

The proof of Lemma 2.5 is complete. \(\square\)

We conclude by stating the dual of Theorem 2.1.

**Theorem 2.1'.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be small finitely cocomplete categories. Then a functor \(G : \mathcal{A} \rightarrow \mathcal{B}\) preserves finite colimits if and only if \(G\) has a right pluri-adjoint. \(\square\)

3. Pluri-adjoints in semilattices

Semilattices provide a particularly simple setting for the concept of pluri-adjoint. For this reason, we found it appropriate to deal separately with this case. One by-product of the treatment of this case is the following conclusion: The 'pluri-Heyting' algebras are precisely the distributive lattices (Corollary 3.4).

We adopt the categorical point of view on posets: A poset is a small category \(\mathcal{C}\) such that, for every two objects \(c, d\), the set \(\mathcal{C}(c, d) \cup \mathcal{C}(d, c)\) has at most one element. If there is such a morphism, say, from \(c\) to \(d\), one writes \(c \leq d\). To say that a poset \(A\) is finitely complete as a category amounts to saying that \(A\) is a meet-semilattice with a last element \(1\) (the meet of the empty set of objects). Since there is at most one morphism between two given objects, all diagrams in such a category commute. A functor \(g : A \rightarrow X\) between two posets thought of as categories is simply an order preserving mapping between the two posets.
A pluri-adjoint for meet-semilattices admits the following simple description, which is a direct translation of Definition 1.1 (For a similar description for adjoints and its different ramifications, the reader is referred to the very instructive paper [18]):

Let $A$ and $X$ be meet-semilattices (with the partial order denoted by $\leq$, the meet by $\wedge$, and the last element by 1) and $g : A \to X$ an order preserving mapping. Then a pluri-adjoint for $g$, viewed as a functor, consists of the following data: For every $x \in X$, there exist a nonempty set $T(x)$ and a family $(f'(x))_{t \in T(x)}$ of elements of $A$ such that $x \leq g(f'(x))$ for each $t \in T(x)$. These data are to satisfy the following properties:

(i) If $a \in A$ and $x \in X$, then $x \leq g(a)$ if and only if there exists at least one $t \in T(x)$ with $f'(x) \leq a$;

(ii) For every $x \in X$, the family $(f'(x))_{t \in T(x)}$ is directed downwards; that is, given $\tau, \rho \in T(x)$, there exists some $\eta \in T(x)$ such that $f'(x) \leq f'(\tau)$ and $f'(x) \leq f'(\rho)$.

For join-semilattices (with a first element, 0), there is a dual characterization for right pluri-adjoints.

We provide the following example of a right pluri-adjoint for semilattices:

Example 3.1. Let $C$ be a complete join-semilattice (that is, arbitrary joins, including 0, exist). An element $a \in C$ is said to be finite (cf. [24, 11.3.1]) if, whenever $a \leq \bigvee_{i \in I} b_i$ for a family $(b_i)_{i \in I}$ of elements of $C$, there is a finite subset $M$ of $I$ such that $a \leq \bigvee_{i \in M} b_i$. Thus, 0 is finite, as one can always take $M = \emptyset$. Also, the join of two finite elements is finite. Thus, the finite elements form a subsemilattice (though not necessarily complete) $F$ of $C$. Let $g : F \to C$ be the inclusion semilattice homomorphism. We will show that $g$ has a right pluri-adjoint: For $x \in C$, let $T(x)$ be the set of all finite elements $y \in C$ such that $y \leq x$. Then $T(x)$ is nonempty, as $0 \in T(x)$. For $\tau \in T(x)$, let $f'(x) = \tau$. Then $g(f'(x)) = f'(x) \leq x$ for all $\tau \in T(x)$. Consequently, if $a \in F$ has the property that $a \leq f'(x)$, then $g(a) = a \leq x$. Conversely, if $g(a) \leq x$ for $a \in F$, then $a \leq x$, and one can find some $\tau \in T(x)$ with $a \leq f'(\tau)$, namely $\tau = a$. On the other hand, if $f'(x)$ and $f'(y)$ are given, then $f'(x) \lor f'(y)$ is finite and it is an upper bound for $f'(x)$ and $f'(y)$. Thus, according to the above characterization of right pluri-adjoints, the above data define a pluri-adjoint for $g$.

The fact that the order preserving mapping (i.e., functor) $g$ defined in Example 3.1 might not have a right adjoint is shown by the following example.

Example 3.2. Let us consider the particular case when $C$ from Example 3.1 is the complete lattice $\mathcal{R}(R)$ of radical ideals of a reduced commutative ring $R$ with 1 (cf. [41]). Then $F$ consists of the radically finitely generated ideals of $R$, that is, those ideals that are radicals of finitely generated ideals. Our example deals with the case when $R$ is the ring $k[x_1, x_2, \ldots, x_n, \ldots]$ of polynomials in countably many indeter-
minates over an algebraically closed field $k$. If $I$ is any radical ideal of $R$, the variety of $I$, denoted $V(I)$, is the set of sequences $(\alpha_i) \in k^N$ such that $h((\alpha_i)) = 0$ for all $h \in I$. It follows immediately that if $I \subseteq J$, then $V(I) \supseteq V(J)$.

Now, suppose that $g$ has a right adjoint $f: C \to F$. This would mean that, for every radical ideal $I$ of $R$, there is a radically finitely generated ideal $f(I)$ with $f(I) \subseteq I$ and such that, whenever $J$ is a radically finitely generated ideal with $J \subseteq I$, it follows that $J \subseteq f(I)$. Let $I$ be the ideal $(x_1, \ldots, x_n, \ldots)$. Since $I$ is prime, it is a radical ideal. Let $f(I) = \text{rad}(h_1, \ldots, h_i)$, where, of course, $h_1, \ldots, h_i$ contain only finitely many indeterminates, say, $x_1, x_2, \ldots, x_n$. Now, let $J = (x_1, x_2, \ldots, x_n, x_{n+1})$ — again, a radical ideal. Since $J \subseteq I$, we have $J \subseteq f(I)$, whence $V(f(I)) \subseteq V(J)$. Since there is no restriction whatsoever on the $(n_0 + 1)$st component of an element from $V(f(I))$, there is some $(\alpha_i) \in V(f(I))$ with $\alpha_{n_0+1} \neq 0$. However, since $(\alpha_i) \in V(J)$, one must have $\alpha_{n_0+1} = 0$. This is a contradiction, and proves that $g$ does not have a right adjoint.

The next result is a semilattice version of our main theorem (Theorem 2.1).

**Theorem 3.3.** Let $A$ and $X$ be meet-semilattices, as above, and let $g: A \to X$ be an order preserving mapping. Then $g$ preserves finite meets (including 1) if and only if $g$, viewed as a functor, has a pluri-adjoint.

Since this theorem is a particular case of Theorem 2.1, the proof will be, for the most part, omitted. Let us only mention that, in order to construct the pluri-adjoint when $g$ preserves finite meets, we proceed as follows: For $x \in X$, the set $T(x)$ can be taken as the set of all finite subsets of $A$. (Thus, in this case, $T(x)$ is the same set for all $x \in X$.) Given $\tau \in T(x)$, $f'(x)$ is defined by $f'(x) = \bigwedge \{b \mid b \in \tau$ and $x \leq g(b)\}$. Since $g$ preserves finite meets, it follows that $g(f'(x)) = \bigwedge \{g(b) \mid b \in \tau$ and $x \leq g(b)\}$. Consequently, we have $x \leq g(f'(x))$. One can then verify the above properties (i) and (ii) of the pluri-adjoint for meet-semilattices.

We note also the dual theorem:

**Theorem 3.3'.** Let $A$ and $X$ be join-semilattices (with 0), and let $g: A \to X$ be an order preserving mapping. Then $g$ preserves finite joins (including 0) if and only if $g$ has a right pluri-adjoint.

We devote the remainder of this section to an application of Theorems 3.3 and 3.3' to distributive lattices.

Recall that a lattice $L$ (with 0 and 1) is said to be distributive if either one of the following equivalent conditions (I) and (II) holds in $L$ [40]: For all $x, y, z \in L$,

(I) \hspace{1cm} x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),

(II) \hspace{1cm} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).

If $a$ is any element of $L$, then the mapping $g'_a: L \to L$ defined by $g'_a(y) = a \wedge y$,
$y \in L$, is order preserving, and hence can be thought of as an endofunctor of the category $L$. Now, to say that condition (I) is satisfied amounts to saying that the functor $g_d$ preserves joins of pairs of elements for every $a \in L$. Since it preserves the 0 anyway, condition (I) amounts, by Theorem 3.3', to the fact that $g_d$ has a right pluri-adjoint for every $a$. Similarly, condition (II) amounts to saying that the functor $g_o : L \to L$ defined by $g_o(y) = a \lor y, y \in L$, preserves finite meets, that is, has a left pluri-adjoint for every $a \in L$.

Thus, we have the following corollary:

**Corollary 3.4.** Let $L$ be a lattice. The following conditions are equivalent:

(a) $L$ is distributive.

(b) The functor $g_o : L \to L$ defined by $g_o(y) = a \lor y, y \in L$, has a (left) pluri-adjoint for every $a \in L$.

(c) The functor $g_o' : L \to L$ defined by $g_o'(y) = a \land y, y \in L$, has a right pluri-adjoint for every $a \in L$. □

Corollary 3.4 asserts that distributivity in a lattice is a pluri-adjoint property. It is an analogue of yet another result due to Freyd (cf. [8, p. 11]), namely, that complete distributivity in a complete lattice is an adjoint functor property. Indeed, a complete lattice $L$ is a *Heyting algebra* if for every $a \in L$, the meet functor $y \mapsto a \land y, y \in L$, has a right adjoint. As a consequence of Freyd’s Adjoint Functor Theorem, this is equivalent to the statement that the complete distributivity condition $x \land (\bigvee_{j \in A} y_j) = \bigvee_{j \in A} (x \land y_j)$ holds for each $x$ and each family $(y_j)_{j \in A}$ of elements of $L$. In light of this fact, the above Corollary 3.4 asserts that a lattice $L$ is distributive if and only if it is a ‘pluri-Heyting algebra’.

Based on the above results, we can provide an example of a functor $G : \mathcal{A} \to \mathcal{B}$ that does not have a pluri-adjoint: It is enough to take $\mathcal{A} = \mathcal{B} - L$, where $L$ is a non-distributive lattice, and for $G$, a suitable join functor $g_d$, as in Corollary 3.4 (cf. [41, Remark 3.5]). Note that $\mathcal{A}$ and $\mathcal{B}$ are finitely complete.

**Acknowledgment**

The authors express their thanks to Professor Michael Barr for his valuable comments.

The authors would also like to express their gratitude to the referee, who made many helpful suggestions.

**References**


