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EPL, 100 (2012) 60003

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J. C. Cressoni$^{1,2}$, L. R. da Silva$^3$, G. M. Viswanathan$^{2,3}$ and M. A. A. da Silva$^1$

$^1$ Departamento de Física e Química, FCFRP, Universidade de São Paulo - 14040-903 Ribeirão Preto, SP, Brazil
$^2$ Instituto de Física, Universidade Federal de Alagoas - Maceió, AL, 57072-970, Brazil
$^3$ Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte Natal, RN, 59078-900, Brazil

received 25 September 2012; accepted in final form 26 November 2012
published online 3 January 2013

PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion
PACS 05.70.Ln – Nonequilibrium and irreversible thermodynamics
PACS 02.50.-r – Probability theory, stochastic processes, and statistics

Abstract – The elephant walk model originally proposed by Schütz and Trimper to investigate non-Markovian processes led to the investigation of a series of other random-walk models. Of these, the best known is the Alzheimer walk model, because it was the first model shown to have amnestically induced persistence — i.e. superdiffusion caused by loss of memory. Here we study the robustness of the Alzheimer walk by adding a memoryless stochastic perturbation. Surprisingly, the solution of the perturbed model can be formally reduced to the solutions of the unperturbed model. Specifically, we give an exact solution of the perturbed model by finding a surjective mapping to the unperturbed model.

Introduction. – The theoretical foundations of the Brownian motion are well known and were laid more than one hundred years ago (see [1–3] and references therein). Since then, the random walk (RW), the simplest realization of the Brownian motion, has been used in the scientific literature as a prototype for modelling applications in various fields. The diffusion of the traditional RW is known as normal diffusion.

Superdiffusion [4–7] is an accelerated diffusion, for which the mean squared displacement grows faster than linearly in time. Theoretical studies of anomalous diffusion are based on generalized Langevin equations [8,9], the fractional Fokker-Planck equation [10,11] and the continuous-time random-walk [12–14] approaches. Superdiffusion is possible only if the necessary and sufficient conditions of the central limit theorem for sums of random variables are not met, otherwise the behavior is always normal diffusion (e.g., Brownian motion). There are two basic mechanisms (not mutually exclusive) by which a random walker can undergo superdiffusion. The first is an infinite second moment for the random-walk step size distribution, as seen in Lévy processes [15–17]. The second mechanism is long-range power law correlations, i.e. long-range memory [18,19].

Stochastic processes with long-range memory are, by definition, non-Markovian, such that master equations and equivalent tools become inadequate to model them. With the aim of trying to gain a deeper understanding of how non-Markovian behavior at the macroscopic level can arise from microscopic stochastic dynamics, Trimper and Schütz [20] proposed what has since become known as the elephant random-walk model. The key new ingredient in the elephant model was full memory of the entire history. A few years later, the Alzheimer walk [21] model was proposed which contained an additional ingredient: the memory was limited to a fraction $f$ of the initial history, rather than the complete history. For $f \to 1$ the Alzheimer walk becomes the elephant walk. Although the Alzheimer walk model led to a deeper understanding of non-Markovian dynamics, the construction of the model is somewhat artificial. Specifically, the degradation of memory, controlled by the parameter $f$, is done in a manner without introducing additional noise. In real systems (e.g., the brain), memory loss is likely accompanied by the introduction of disorder, i.e. new sources of randomness. Here we study a modification of the Alzheimer walk, by adding a source of uncorrelated, i.e. memoryless, noise. The surprising finding which we report...
here is that the exact solutions of the modified model map back to those of the Alzheimer walk model. Specifically, when the Alzheimer walk model is perturbed by the addition of uncorrelated noise, the resulting model is identical to some other solution of the unperturbed Alzheimer walk model. In this sense, the Alzheimer walk model is robust under stochastic perturbations.

We first briefly describe our approach in general terms. Three essential ingredients for the understanding of superdiffusion, namely, long-range memory correlations, randomicity and memory damage, are introduced in the Alzheimer walk non-Markovian model. The long-range memory is allowed to range from full memory, through mid-range size memory of length \( ft \), to the zero-length memory size corresponding to the single initial step. The new source of disorder is completely uncorrelated noise. We obtain analytical solutions that allow the characterization of all possible diffusion regimes and the displaying of all the diffusion phases. The properties of the phases include log-periodic oscillations that appear for small sizes of the long-range memory. Log-periodic oscillations in RW have been reported to appear elsewhere (see e.g., [21–23]). We also show that the size of the region of the phase diagram with superdiffusion is controlled by the memoryless noise.

The onset of superdiffusion in the elephant and Alzheimer walk models has been shown to be related to a tendency to repeat [20] or negate [21] previous actions of the walker. The former leads to classical superdiffusion while the latter produces log-periodic superdiffusion. In both these models, these are the only two options available to the walker. Either the walker repeats some previous step or else takes the opposite step. Disorder or noise enters the picture only via the random choice of the previous time (for repeating or negating what happened at that earlier time).

But what happens if we now introduce noise directly into the dynamics, as is usually done in the Langevin equation approach? The addition of noise in stochastic studies has been vastly used, e.g., in biology [24–26]. We implement this noise by allowing the walker, with probability \( s \), to move independently of the past. Allowing the walker to “mix” random decisions with decisions based on past memories takes us one step closer to the real world. Animal motion might be a combination of these two ingredients. Chemically marked trails, for example, like ant networks of paths, are constructed with evaporative scent markers (pheromones). The track can, accordingly, disappear in some regions leaving unmarked areas along the path network. In such places, the individual following the track must move without reference to the past, i.e., (more) randomly. Until a track is found again by the insect, random movement may be the best (or only) option.

Below we introduce the model, report our findings, and conclude with a discussion of our results.

**Model and methods.** — The model we describe in this work generalizes the one-dimensional elephant walk (EW) [20] in which the decisions at present time are all based on the actions. Within the model, a new stochastic parameter is introduced, drawn from a uniform distribution, which accounts for randomness in the walk. The walk is therefore a mixing of steps taken with basis on well founded decisions and random steps. It is a traditional random walk that has also the ability to recall its past actions and accept or refuse them.

This work should be contrasted with a newly published model [27], which also generalizes the EW model, by introducing stop points in the walk. The model is exactly solved and presents subdiffusion.

Within the mixed random elephant walk model, at any time \( t \), the decision to move at time \( t \) is taken from three random variables, i.e., \( p + q + s = 1 \). The walker can accept a past decision at time \( t' < t \), chosen from a uniform distribution, with probability \( p \), take the opposite action with probability \( q \), or take a random step with probability \( s \).

Notice that the steps taken, whatever based on past actions or random, are all part of the memory as a whole. This meaning that, if at time \( t \) the walker decides to refer to its past memories to decide what to do, and the selected past time is \( t' \), then the action taken at \( t' \) will be accepted (refused) with probability \( p(q) \). It does not matter if at \( t' \) the decision was random or memory based.

The elephant walk can be recovered for \( s = 0 \) and the traditional random walk is recovered for \( s = 1 \). In this way, \( s \) is a key parameter controlling the amount of randomness in the model. Notice also that in some sense the non-Markovianity of the model is also related to \( s \).

The model describes the motion of a random walker with equal time steps in one dimension. In each time step, the walker currently at a position \( X_t \), moves either one step, right or left, to \( X_{t+1} \). The choice to move randomly or refer to past memories is decided according to a variable \( s \), which is randomly chosen from a uniform distribution. The decision to move at random is then accepted with probability \( s \). If the decision to move at random wins, the walker takes a step to the right or left with equal probability. Otherwise, a previous time \( t' \) between 1 and \( t \) is selected from a uniform distribution. The past action taken at time \( t' \) is then accepted with probability \( p \) or refused with probability \( q \). Notice that the part of the walker that consults the past memory is similar to the what happens in the elephant model. According to this algorithm, the random steps can happen if: i) the decision to move randomly is decided by the value drawn for \( s \) or if \( t' \) was a random step.

The probabilistic equations can be easily written down. The walker moves according to \( p + q + s = 1 \). We can now set the initial condition. At time \( t = 0 \) the walker always takes a right step, i.e., \( X_1 = 1 \). It is important to have a deterministic initial condition in order to solve the Fokker-Planck equation of the elephant model [20].
The past history of the walker decisions is taken into account by randomly selecting a previous 1 ≤ \( t' \leq ft \) drawn from a uniform distribution. A random variable \( \sigma_t = ± 1 \) is then defined, based on the value of \( \sigma_{t'} \) in the following manner:

\[
\sigma_t = \begin{cases} 
+\sigma_{t'}, & \text{with probability } p, \\
-\sigma_{t'}, & \text{with probability } q.
\end{cases}
\]  

(1)

Another random variable \( \tau_t = ± 1 \) is also defined with equal probability \( a \text{ priori} \). The probabilistic recurrence relation can then be written as

\[
X_{t+1} = X_t + \begin{cases} 
\sigma_{t+1}, & \text{with probability } p + q, \\
\tau_{t+1}, & \text{with probability } s.
\end{cases}
\]  

(2)

Without loss of generality we assume that the first step always goes to the right, \( i.e., \sigma_0 = +1 \). The position at time \( t \) is simply given by \( X_t = \sum \sigma_t + \sum \tau_t = \sum \sigma_t \), since the overall sum in \( \tau_t \) is null and for the sake of simplicity we assume that \( X_0 = 0 \) for \( t = 0 \). Notice that \( \sigma_t = ± 1 \) and \( \tau_t = ± 1 \) represent random numbers, associated with the long-range memory and random motion, respectively.

They represent stochastic noises: \( \sigma \) contains two-point correlations, or memory, whereas \( \tau \) represents a white noise with no correlations.

The memory length \( L_f \) depends on a parameter \( 0 \leq f \leq 1 \). For \( f < 1 \) we can write \( L_f \equiv 1 + \lfloor ft \rfloor \), where \( \lfloor ft \rfloor \) denotes the largest integer smaller than \( ft \). For \( f = 1 \) the memory size is \( L_{(f=1)} = t \), thereby recovering the full memory history of the walker. The memory size then ranges from 1 to \( L_f \), \( i.e., t' \in [1, L_f] \). In the asymptotic limit \( t \to \infty \) we shall write \( L_f \equiv L = ft \).

The model described above can be exactly solved. In fact, it is easy to map the model onto the elephant model, which has an exact analytic solution [20]. As stated above, the parameters obey the probabilistic normalization condition \( p + q + s = 1 \). The purely random part of this equation comes from the stochastic variable \( s \), which is associated with a right or left step with equal probability \( a \text{ priori} \). This is equivalent to look at the action taken at any past time \( t' \) and accept (or refuse) it with probability 1/2. Therefore the probabilistic variables can be regrouped as \( (p + s/2) + (q + s/2) = 1 \). The part that accepts the action taken at \( t' \), \( i.e., s/2 \), is joined to \( p \). Similarly, the part that refuses the past action at \( t' \) is joined to \( q \).

We now call \( p' = p + s/2 \) and \( q' = q + s/2 \) and write a new probabilistic equation as \( p' + q' = 1 \). With this simple maneuver, we recover the elephant model and all solutions already available for that model. A more rigorous proof that this mapping is feasible can be obtained by considering the probabilistic normalization equation. We consider the effective probability of moving to the right, \( P_{eff} \), which, in terms of \( (p, q, s) \) is given by

\[
P_{eff}^+(p, q, s) = \frac{n^-}{t}p + \frac{n^+}{t}q + \frac{s}{2}.
\]  

(3)

where \( n^+ (n^-) \) represents the total number of right (left) steps taken by the walker up to time \( t \). A similar equation can be written for the effective probability of moving to the left, \( i.e., \)

\[
P_{eff}^-(p, q, s) = \frac{n^-}{t}p + \frac{n^+}{t}q + \frac{s}{2}.
\]  

(4)

It is easy to check that \( P_{eff}^+(p, q, s) + P_{eff}^-(p, q, s) = 1 \), since \( n^+ + n^- = t \) and \( p + q + s = 1 \). We can therefore write \( (3) \) as

\[
P_{eff}^+(p, q, s) = \frac{n^+}{t}p + \frac{n^-}{t}q + \frac{s}{2}
\]  

(5)

\[
= \frac{n^+}{t}p + \frac{n^-}{t}q + \left( \frac{n^+}{t} + \frac{n^-}{t} \right) \frac{s}{2}
\]  

(6)

\[
= \frac{n^+}{t}p + \frac{n^-}{t}q + \frac{s}{2}
\]  

(7)

Since, on average, half of the steps due to the choice of \( s \) represent right steps, we have joined \( s/2 \) to \( p \), and similarly for \( q \). If now define \( p' = p + s/2 \) \( (q' = q + s/2) \) as the probability of moving to the right (left), we can write

\[
P_{eff}^+(p', q') = P_{eff}^+(p, q, s).
\]

Since the normalization condition can be written in terms of \( (p, q, s) \) or \( (p', q') \), we can map the original model onto a model controlled only by the random variables \( (p', q') \). Therefore, the same equations derived for the Alzheimer walk model [28] can be used to analyze the present model. In what follows we shall briefly present the main points already discussed in the above paper.

Results and discussions. – The first moment \( \langle x_t \rangle \) is central to our discussions for several reasons. The asymptotic coefficient can be used to determine the Hurst exponent \( H \) and consequently the critical lines separating the diffusion regimes. Besides, it is important to identify the log-periodic and escape regimes in the asymptotic limit \( t \to \infty \). We discuss the first moment in terms of the parameters \( \beta = 2p - 1 \) and \( f \), analogously to the notation used in previous papers (see, \( e.g., [28] \)).

In the asymptotic limit we can write an equation for the first moment, \( i.e., \), \( \frac{d\langle x_t \rangle}{dt} = \sigma_{eff} = \beta(x_L - x_0)/L \). Thus, considering \( x_0 = 0 \), one finds the differential equation for the first moment,

\[
\frac{d}{dt}\langle x_t \rangle = \frac{\beta}{ft}\langle x_{ft} \rangle.
\]  

(8)

valid in the asymptotic limit. This equation has a general solution in terms of an expansion of the form \( \langle x_t \rangle = \sum A_i t^{d_i} \sin[B_i \ln(t) + C_i] \sim At^{d_0} \sin[B_0 \ln(t) + C], \) where only the leading term survives as \( t \to \infty \). Inserting this back into eq. \( (8) \) leads to a system of transcendental equations for the leading pair \( (\delta, B) \), \( i.e., \)

\[
\delta = \beta t^{d_0 - 1} \cos(B \ln f),
\]  

(9a)

\[
B = \beta t^{d_0 - 1} \sin(B \ln f).
\]  

(9b)

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This set of equations is used to draw the lines separating the different regimes in the two-dimensional phase diagram in the \((\beta, f)\)-or \((p', f)\)-plane. We start by separating the log-periodic and non-log-periodic regimes which occurs in the \(\beta < 0\) region. Oscillating solutions only exist for non-zero values of \(B\). The oscillation threshold is found setting \(B = 0\), which gives a critical line dividing the oscillatory and non-oscillatory regimes. Equation (9b) is immediately satisfied for \(B = 0\) while eq. (9a) leads to

\[ \delta = \beta f^{\delta - 1} \]

with solutions with \(B = 0\) only within the region defined by \(f > f_0(\beta)\) where \(f_0\) is written in terms of the Lambert \(W\)-function [29], i.e.,

\[ f_0(\beta) = c|\beta|W\left(\frac{1}{c|\beta|}\right) \]

(see appendix A in ref. [29] for details). The pairs \((\beta, f)\) within the remaining region in the \((\beta, f)\) phase diagram, obeying \(f < f_0(\beta)\), are compatible with non-zero values of \(B\). The asymptotic solutions for the first moment within this region are therefore characterized by log-periodic oscillations. The critical line obtained by fixing \(f = f_0(\beta)\) marking the onset of log-periodicity is then given by (11) which is represented by a dashed line in fig. 1(a)–(d).

We now look to separate the normal and anomalous phases. The separation between normal and superdiffusive regimes can be obtained from the value of \(\delta\). We have argued before [29] that the diffusion behavior is normal (Hurst exponent \(H = 1/2\)) if \(\delta < 1/2\). On the other hand, we have superdiffusion (Hurst exponent \(H = \delta\)) if \(\delta > 1/2\) (actually it is only marginally superdiffusive for \(\delta = 1/2\)). For \(\beta < 0\) we set \(\delta = 1/2\) and dispose of \(B\) in eqs. (9a), (9b). This gives a critical line, i.e.,

\[ 2\sqrt{\frac{\beta^2}{f_c} - \frac{1}{4}} = \tan \left[ \ln(f_c) \sqrt{\frac{\beta^2}{f_c} - \frac{1}{4}} \right], \]

which separates regions I and II in fig. 1(a)–(d). For now on, for simplicity, we shall avoid using indexes to represent the critical values of the parameters. For \(\beta > 0\) we have to set \(\delta = 1/2\) and \(B = 0\) in eqs. (9a), (9b). This corresponds to set \(\delta = 1/2\) in eq. (10), which gives

\[ f = 4\beta^2 \]

representing the critical line separating the diffusive (V) and the anomalous (VI) regimes in fig. 1(a)–(d). Note that \(f = 1\) gives \(\beta = 1/2\) (or \(p' = 3/4\)) in agreement with ref. [20].

There are other regimes for \(\delta < 1/2\), leaving a total of six different phases. All different regions are listed in table 1. We call attention for the normal diffusive regions with positive \(\delta\) (regions II and V). In this case the first moment diverges despite \(H = 1/2\). These regimes are called escape regimes. The critical lines (between regions II and III for \(\beta < 0\) and regions IV and V for \(\beta > 0\)) are obtained from eqs. (9a), (9b) with \(\delta = 0\). For \(B = 0\) only eq. (9a) is needed leading to (10), i.e., \(\delta = \beta f^{\delta - 1}\). For \(\delta = 0\) this gives \(\beta = 0\) or \(p' = 1/2\). This is represented by a vertical line in the \((f, p')\)-plane separating regions IV and V. Notice that \(\delta > 0\) in region V, which means that the first moment diverges in the asymptotic limit. This region corresponds to a escape regime. On the other hand, region IV corresponds to \(\delta < 0\) and therefore the regime is normal. The critical line separating regions IV and V is simply given by \(\beta = 0\) (or \(p' = 1/2\)). For \(B \neq 0\) we are now within the log-periodic region. In this case \(\delta = 0\) in eq. (9a) gives \(\cos(B \ln f) = 0\) or \(-B \ln f = (\pi/2)(2n + 1)\). This equation has meaningful solutions for \(n = 0\), or \(-B \ln f = \pi/2\). Therefore sin\((B \ln f) = -1\) in eq. (9b). Then \(B = -\beta/f\) and we are left with

\[ \beta \ln f = (\pi/2)f, \]

which is the critical line separating regions II and III.

Table 1 lists all phases along with the critical separation lines and the corresponding values for \(\delta\) and the Hurst
A similar table was shown in a previous work [28] for a model without the random parameter $f$, $p$-plane for several values of the parameter $f$. We see that for small values of $f$ and randomization from table 1 by fixing the corresponding value of $f$ for several values of the parameter $f$. The separation lines in $p,s$ are obtained for several values of $f$. The lines separating the different phases as a function of $p,s$ can be derived from table 1 by fixing the suitable value for $f$. The dot-dashed lines represent the maximum possible value of $p$ which is given by $p_{max} + s = 1 (p + q + s = 1$ for $q = 0)$. The region above the dot-dashed line is forbidden. Anomalous diffusion do not exist for $s > s^*$. This is clearly shown in fig. 2(d) where the locus of $(s^*, f^*)$ is shown. The equation for this line is obtained from eq. (13) with $\beta = 2p + s - 1$ and $p^* + s^* = 1 (p^* obeyes the same equation as $p_{max}$) and is given by $f^* = 4(1-s^*)^2$. Anomalous diffusion does not exist above this line. Figure 2(d) shows the locus of $(s^*, f^*)$. The equation for this line is given by $f = 4(1-s^*)^2$. Normal or superdiffusion can exist on the region to the left of this line, but only normal diffusion exists to the right of the line. The effects of the two competing parameters $f$ (memory damage, small $f$ favors superdiffusion) and $s$ (random motion, large $s$ favors normal diffusion) on the model are clearly seen from this panel.

**Conclusions.** – We consider a model that introduces random perturbations to the Alzheimer walk model in the form of uncorrelated noise. The exact solutions are exhibited that lead to the diffusion phases of the model. The stochastic perturbations enrich the walk by introducing randomness to the memory, therefore leaving room for studying the effect of memory lapses, as expected to occur in real systems. It was shown that the perturbations bear...
a strong relation to the superdiffusion properties of the model. For large perturbations, anomalous diffusion can be completely removed from the system, even in the presence of long-memory correlations. One important aspect of this work is that the solutions of the memoryless perturbation model can be exactly mapped onto the unperturbed model. We hope that these results can be useful for the study of real systems in the future.

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We acknowledge CNPq and FAPESP (grants numbers: 2011/13685-6 (JCC) and 2011/06757-0 (MAAS)) for financial assistance. GMV also thanks FAPEAL.

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