Dissipative Lévy random searches: Universal behavior at low target density

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I. INTRODUCTION

Random search models are amenable to describe a great diversity of phenomena [1], ranging from the biological context, including animal foraging [1–5], regulatory proteins looking for resources [6], and the finding of binding sites on proteins by neurotransmitters in the brain [7], to industry and automated computer searchers of registers in high-capacity databases [8]. In most cases, the goal to accomplish (i.e., the general finding of targets) also brings about some inherent mechanism of dissipation (e.g., the energy cost while foraging for resources). This subtle balance between the “energy” intake and search cost makes the choice of the appropriate search strategy crucial in adverse situations with shortage of targets. In fact, a suitable choice might set the distinction between an irreversible failure process and a successful one. In animal foraging, for example, a proper strategy can determine the survival of the species in a meagre environment [9].

These competing mechanisms set the scenario for the emergence of a dynamics phase transition [10] from an active to an irreversible absorbing state in scarce landscapes. By considering a searcher with a power-law Lévy distribution of step lengths, here we calculate analytically and numerically the critical exponents, critical density of targets, average energy gain, and survival rate as a function of the search dynamics (ballistic, superdiffusive, or Brownian). When the target density is critically low, we find that the searcher should increase as much as possible its diffusiveness as a survivor strategy. Our results evidence the universal aspects of dissipative searches. Indeed, as the critical exponents remain robust with respect to the dynamics of the search and distinct mechanisms of dissipation, these critical search processes are found to belong to the same universality class.

II. AVERAGE ENERGY GAIN

We start by considering a random searcher looking for static target sites in a one-dimensional search space (see Fig. 1). The searcher performs steps of length drawn from a power-law probability density function (pdf) $p(\ell_i) \propto |\ell_i|^{-\mu}$, if $|\ell_i| \geq \delta_0$, valid for all steps $i$. For $1 < \mu < 3$, this pdf corresponds [12] to the long-range asymptotical limit of the $\alpha$-stable Lévy distributions with superdiffusive dynamics ($\mu \to 1$ sets the ballistic limit). On the other hand, the diffusive (Brownian) regime occurs for $\mu > 3$. The cumulated energy [9] of a given walk after $N$ steps can be generally written as

$$E_N = \sum_{i=1}^{N} [g_i \delta_i - f(|\ell_i|)],$$

(1)

where $\delta_i = 1$ ($\delta_i = 0$) if a target is found (not found) at the $i$th step, with energy intake $g_i > 0$, and $f(|\ell_i|) > 0$ is a dissipation cost function generally dependent on the step length. The energy balance between the income and dissipative terms in Eq. (1) allows a dynamics phase transition between limit situations in which the searcher remains in the active phase indefinitely ($E_N > 0$ for any $N$) or enters an irreversible absorbing state when $E_N = 0$, in the scarce regime of resources in which nearby targets are rarely available on average [11].

Consider now a set of search walks in which $n$ targets are found, irrespective of the number $N$ of steps performed. Every time a target is found, its content (in energy, general units of accomplished goal, etc.) is completely absorbed and a new target is generated. In order to keep the density of targets $\rho_h$ unaltered, the distance $\lambda \propto \rho_h^{-1}$ between the two closest targets...
(right and left boundaries) is kept fixed. The \( n \) searches of each walk are assigned to a given sequence \( \{x_j\} = \{x_1, x_2, \ldots, x_n\} \) of initial distances to the left target before each encounter. Here, the starting positions are taken from a \( j \)-independent pdf \( \pi(x_{j+1}) \) and the finding of the right target happens after four steps.

FIG. 2. (Color online) Search model: in this example, in its \( j \)th search the searcher starts from \( x_j \) and finds the left target after three steps. In the sequence, a new starting position \( x_{j+1} \) is drawn from the \( j \)-independent pdf \( \pi(x_{j+1}) \) and the finding of the right target happens after four steps.

\[
\langle L \rangle \approx 3x_j + 5 \lambda,
\]

where we took \( g = g \), and \( \langle L \rangle(x_j) \) is the mean distance traversed between consecutive encounters. The exact calculation of \( \langle L \rangle \) yields [13] \( \langle L \rangle(x_j) = [\langle x \rangle - \lambda - L^{-1} \langle \ell \rangle(x_j) \rangle, \) with the integral operator \( \mathcal{L} f(x) = \int_{r_1}^{x - r_1} p(x - x') f(x') dx' \), where \( r_i \) is the searcher’s perceptive distance, \( \mathcal{L} \) is the unity operator, and \( \langle \ell \rangle(x_j) \) is the mean value of the single step starting from \( x \). Through a discretization procedure of the search space [13], \( \langle L \rangle(x_j) \) can be calculated, with the results shown in Fig. 2.

Alternatively, we can also write \( \langle L \rangle(x_j) \approx \langle N \rangle \langle \ell \rangle \), where \( \langle N \rangle = [\langle x \rangle - \lambda - L^{-1} \langle \ell \rangle(x_j) \rangle, \) with \( h(x) = 1 \), denotes the mean number of steps between consecutive encounters [13]. For the power-law Lévy pdf, a continuous time approximation [13] leads to \( \langle N \rangle(x_j) = C(\lambda - x_j - r_0)^{\beta}, \) with \( C(\lambda, \mu) = A \sin(\pi \tilde{\mu})/(\pi \tilde{\mu} \xi_0^{\tilde{\mu} / 2}), \) \( \tilde{\mu} = (\mu - 1)/2 \) and \( A(\lambda, \mu) = 1 \) if \( \mu \leq 2 \). As seen in Fig. 2, this approximation actually compares nicely with the exact result for \( \langle L \rangle \).

FIG. 3. (Color online) Normalized average energy gain \( \psi \) versus energy density \( \rho / (\lambda - 2r_0) \), for \( n = 100, \rho = \ell_0 = \alpha = 1 \), and (left to right) \( \mu = 1.1, 1.5, 2.0, 2.5, 2.9 \). Numerical results averaged over 5 \( \times 10^6 \) walks are shown in solid blue squares and empty green circles for possible absorption of the searcher only after \( n \) encounters or at any \( j \leq n \), respectively. Analytical results from Eq. (3), shown in black lines, nicely agree with the former case. Insets: critical behavior of \( \psi \) in the former case, with best fits shown in red lines leading to \( \beta \).

We next perform the average over the ensemble \( \{x_j\} \) of initial distances drawn from the pdf \( \pi(x_j) \). The average net energy of the searcher after \( n \) encounters (normalized by \( n \alpha \)) is thus given by \( \psi(\lambda, \mu) = \langle E_n \rangle/(n \alpha) = g/\alpha - m(\lambda, \mu), \) where \( m(\lambda, \mu) = \langle L \rangle = 2 \int_{r_0}^{x_1} \langle L \rangle(x) \pi(x) dx \). This result readily leads to the identification of the critical inverse density of targets, \( \lambda = \lambda_c \), for which the average energy is null: \( m(\lambda_c, \mu) = g/\alpha \). As mentioned, for lower target densities, \( \lambda > \lambda_c \), the searcher is at the absorbing state, with \( \psi = 0 \), whereas in the active state its net average energy is positive, i.e., \( \psi > 0 \) for \( \lambda < \lambda_c \). In this sense, \( \psi \) can be characterized as an order parameter of this absorbing phase transition.

Figure 3 displays \( \psi \) as a function of the characteristic energy density, \( \rho = g / (\lambda - 2r_0) \), for the case of a homogeneous pdf of initial distances, \( \pi(x_j) = (\lambda - 2r_0)^{-1}: \)

\[
\psi(\rho, \mu) = \frac{g}{\alpha} - \frac{2\pi \tilde{\mu} \tilde{\mu} - 3 \xi_0^{2\tilde{\mu}} \rho}{8(2 - \mu) \rho \cos(\pi \mu / 2)} + \frac{2\sqrt{\pi \tilde{\mu} \Gamma \xi_0} \rho^{2\tilde{\mu}}}{2^{\mu - \mu} \rho^{2\tilde{\mu}}},
\]

(3)
with the ratio between the gamma functions given by $\bar{\Gamma}(\mu) = \Gamma(1/2 + \mu/2)/\Gamma(1 + \mu/2)$. Numerical results are also shown for comparison in situations in which the searcher’s absorption can occur only after $n$ encounters or at any $j \leq n$. We note a nice agreement between the analytical and numerical results in the former case. The numerical $\psi$ curves of the latter case lie below due to the walks that are absorbed earlier. In particular, we also observe in all cases the expected saturation of the normalized $\psi$ function in high densities: $\psi \rightarrow g/\alpha$ for $\rho \gg \rho_c$. A remarkable agreement between the analytical and numerical critical values occurs for all $\mu$ (see Table 1). As the active phase is present in a wider range of target densities when $\mu \rightarrow 1$, thus a faster (ballistic) strategy becomes advantageous in order to access faraway targets in critically low densities. Indeed, searches with $\mu \gtrsim 1.5$ are already in the absorbed state ($\psi = 0$) for $\rho \gtrsim 0.7$, whereas for $\mu \rightarrow 1$ $\psi$ becomes null only at $\rho_c \approx 0.5$.

In the close vicinity of the critical point, i.e., for $\epsilon = x_c - x = (\lambda_c - 2\epsilon_c)/g - (\lambda - 2\epsilon)/g \rightarrow 0$, Eq. (3) leads to

$$
\psi(\epsilon, \mu) \approx \frac{2\pi \bar{\mu}(\mu - 3)C_\epsilon C_\epsilon^{2/3} g}{8(2 - \mu) \cos(\pi\mu/2)} \frac{4\sqrt{2\pi \bar{\mu}^2 C_\epsilon \delta E_\epsilon^{2/3}}}{2\mu(2 - \mu)\rho_c^{-2}} e^{\beta^*},
$$

with $C_\epsilon = C(x_c, \mu)$ and the $\mu$-independent critical exponent $\beta^* = 1$. As shown in the inset of Fig. 3, this finding nicely agrees with the estimate $\beta^* \approx 1$ from the nonlinear best fitting of the numerical results in the whole range $1 < \mu < 3$. The prefactor in Eq. (4) is larger for smaller $\mu$, indicating a larger energy gain for searches with faster dynamics at the same distance $\epsilon$ to the critical point. It is also interesting to contrast this finding with the exponent $\beta^H \approx 2$ of searches in which the absorption can occur at any $j \leq n$. Actually, in this case, in addition to the probability to have null energy as $\epsilon \rightarrow 0$, one has also to consider the survival rate of the searcher up to $n$. As these two functions present similar $n$ critical behavior (see below), the result $\beta^* = 2\beta^H$ thus arises. Nevertheless, their values of $\chi_c$ are the same for a given $\mu$. These findings are also corroborated below by the more sensitive finite-size scaling analysis.

III. SURVIVAL RATE

The survival rate $\Gamma(n, \lambda)$ measures the probability of the searcher to remain in the active state after $n$ encounters. To calculate $\Gamma$, we first notice that Eq. (2) can be written in the form $\langle L_n \rangle = \sum_{j=1}^{n} E_j$, with $E_j = g - \alpha(L)g(x_j)$ representing the net energy gain in the $j$th encounter. In this sense, a suitable mapping can be established with the position of a standard random walker after a sequence of $n$ steps of lengths $|E_j|$. As $E_j$ depends on $x_j$ through $L(L(x_j))$, the pdf of “energy step lengths” can be readily obtained: $\omega(E_j) = 2\pi(x_j)|dL(L)/dx_j|^{-1}|dE_j/dL|^{-1}$.

Now, by defining $P(E, n)$ as the probability of the searcher to accumulate net energy $E$ after $n$ encounters, we may ask about the probability $F(E, n)$ of the first passage through the value $E$ after $n$ encounters. Contributions to $P(E, n)$ can come from walks that have reached net energy $E$ for the first time at some earlier number of encounter $n' < n$. Subsequently, a multiplicity of possible returns to $E$ in $n - n'$ encounters exists so that $P(E, n) = \sum_{n' < n} F(E, n')P(E, n - n')$. The first term takes care of the initial search condition through the Kronecker deltas. The standard way to solve this equation involves [14] the definition of the generating functions, $P(E, z) = \sum_{n=0}^{\infty} P(E, n)z^n$ and $F(E, z) = \sum_{n=0}^{\infty} F(E, n)z^n$, leading to $F(0, z) = 1 - 1/P(0, z)$. The above mapping combined with the central limit theorem that drives the pdf $\omega(E_j)$ allows us to write $P(E, n)$ as a Gaussian function with $\Gamma(n, \epsilon) = n\epsilon - \alpha - \delta E$ and square deviation $\sigma^2 = n(\epsilon - \alpha)^2$. In this context, we note that $\Gamma$ nullifies in the absorbing phase, whereas $\Gamma > 0$ in the active state, with the expected saturation ($\Gamma \rightarrow 1$) in the high-density regime. These features also confer to the survival rate the status of an order parameter of the absorbing phase transition [10]. Below we apply scaling ideas near the critical point to check on the above asymptotic prediction for $\Gamma$.

IV. FINITE SIZE SCALING

We assume the following scaling forms as $\epsilon \rightarrow 0$ [10]: $P(\epsilon, n) \sim n^{-\beta/\nu} f_\psi(n^{1/\nu})$ and $\Gamma(\epsilon, n) \sim n^{-\beta/\nu} f_\Gamma(n^{1/\nu})$, with the respective scaling functions $f_\psi$ and $f_\Gamma$. In this context, the critical exponents can be obtained by considering the functions $\psi$ and $\Gamma$ at several values (size scales) of $n$, along
with the use of auxiliary functions. For example, by defining $k(n,n_0,\varepsilon) = \ln(\Gamma(n,\varepsilon)/\Gamma(n_0,\varepsilon))/\ln(n/n_0)$, we notice at the critical point that $k(n,n_0,0) = -\beta/\nu$. As the numerical $k$ curves for a fixed $n_0$ and distinct $n$ cross at the critical point, this procedure also offers an independent way to calculate $\chi_c$. On the other hand, the exponent $\nu$ is found through the definition [9] of $h(n,n_0,\varepsilon) = \partial_\varepsilon \ln(\Gamma(n,\varepsilon))/\partial_\varepsilon \ln(\Gamma(n_0,\varepsilon))$ via $\ln(h(n,n_0,0)) = \nu^{-1} \ln(n/n_0)$. In Figs. 4(b) and 4(c) we display functions $k$ and $h$ for $\mu = 2$ and several $\chi$ and $n$.

A summary of the critical exponents is presented in Table I. As mentioned, the calculation of the exponents regard absorption processes after $n$ encounters, with the exception of $\beta^*$, which is associated with absorption that can occur at any $j \leq n$. We have checked that the exponents remained essentially unaltered for the choices $n_0 = 100$ and $n_0 = 400$. A nice agreement between the analytical and numerical exponents is observed, as well as a remarkable concordance between the analytical and numerical critical densities. It is interesting to observe that $\beta$ presents the mean-field-like value of absorbing phase transitions [10]. In spite of this, the asymptotic $n \gg 1$ behavior of the survival rate, $\Gamma(n,\varepsilon) \sim (\sigma n)^{-1/2}$, typical of first passage time problems [14], implies $\beta/\nu = 1/2$, so that the value $\nu = 2$ emerges in accordance with the numerical estimate. Furthermore, the exponents have not shown any detectable dependence on $\mu$. In spite of this, an important dependence of $\chi_c$ on $\mu$ is observed, with searches of faster dynamics (smaller $\mu$) presenting lower critical density.

We also observe that these sets of exponents differ from those obtained in a numerical study [9] of dissipative random searches with a distinct boundary condition on the end of the walks. In that case, by setting the end of the search to the completion of a fixed total distance traversed, the statistical averages were performed over searches with a distinct number of encounters. This procedure influences the long-term dynamics of the search and the associated exponents. Indeed, it often includes in the averages walks that terminate not due to the finding of a target nor to transition to the absorbing state, but just due to fulfillment of the displacement criterion.

VI. UNIVERSALITY

Beyond the independence of the critical exponents on $\mu$, we have also checked on their robustness with respect to the functional form of the dissipation function. On the analytical side, by replacing $\alpha(L)$ by a more general cost function in Eq. (2), $f(L) = \sum_{k=0}^{\infty} a_k L^k$, the critical exponents actually kept unaltered. In particular, the critical behavior of the average net energy becomes $\psi(\varepsilon,\mu) \sim \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} B_k C_k \eta^j \varepsilon^{\beta}$, with $\beta = 1$ as well, $B_k = a_k [A0 \sin(\pi \mu)/(2\pi(2-\mu)\varepsilon)]^k$, $C_k = \varepsilon^k (2-\mu)^{k-1} k!(2-2\mu)^{k-1}/[k!(k-l)!]$; $C_k = l!/l!(l-j)!\Gamma(l)!\Gamma(m)!(m+i(2-\mu))/[\Gamma(2+k(\mu-1)+i(2-\mu))\varepsilon^k]^{i(2-\mu)}$, and $m = 1 + k(\mu-1)/2 + j(2-\mu)$.

On the other hand, finite-size scaling analysis applied with the use of a quadratic cost function, $f(L) = \gamma |L|^2$, instead of a linear one, leads numerically to the same exponents, though the critical densities vary, as expected. These universality findings reveal that random searches with distinct mechanisms of dissipation share the same critical properties in absorbing phase transitions.

Further, we have also checked on the universality of our results by considering other sets of parameters: $g = 300$ and $\alpha = 1; g = 300$ and $\alpha = 2; g = 100$ and $\alpha = 3; g = 400$ and $\alpha = 3$. In all cases the critical point varies, as expected, but the set of critical exponents remains essentially unaltered.

Even more strongly, these results seem also to hold independently of the details of the search process. For example, the same exponents are numerically found with presence of movable targets performing Lévy walks.

VI. DISCUSSION AND CONCLUSIONS

The choice of the appropriate random search strategy is crucial in adverse situations in which the density of target sites is critically low. In particular, if the search involves some mechanism of dissipation, a suitable choice might be the difference between an irreversible failure process and a successful one. This sort of idea actually finds application in a diversity of relevant contexts ranging from biological phenomena [1–7] to automated computer searchers [8]. In animal foraging, e.g., a proper strategy can make the difference between the extinction and survival of the species.

In this work, we have applied methods and concepts of dynamics phase transitions to study analytically and numerically how a general dissipative mechanism can drive a Lévy random searcher to a transition from an active to an irreversible absorbing state in shortage of targets.

We have observed that ballistic strategies allow the searcher to remain in the active state in a wider range of search landscapes. In other words, smaller values of the Lévy index $\mu$ lead to lower critical densities. Interestingly, the general properties of the random search near the critical point are driven by critical exponents which do not depend on the dynamics of the search ($\mu$). In spite of this, the advantage of a smaller $\mu$ is also present in the energy gain through a higher prefactor.

All these features impact considerably the dynamics of the search in critically scarce environments. Actually, when resources are low, the searcher (e.g., an animal in the foraging activity) should increase as much as possible its diffusiveness as a survivor strategy to reach faraway targets. Alternatively, since critical phenomena involve fluctuations of several magnitudes, a small bias toward the positive trend (e.g., a change to a slightly richer environment) might guarantee the survival of the species in very adverse conditions [15].

Finally, the robustness of our results on the critical exponents applies not only with regard to the search dynamics but also to the specific form of the cost function. Indeed, we have shown that the absorbing phase transitions of random searches with distinct mechanisms of dissipation actually belong to the same universality class. This surprising result implies a universal behavior near the critical survival condition, independent of the details of the search process.

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DISSIPATIVE LÉVY RANDOM SEARCHES: UNIVERSAL ... by G. C. van der Veer and C. Gale (ACM, New York, 1995).
[11] This contrasts with the so-called nondestructive search [2], in which nearby targets (e.g., the last target found) are always available to the searcher.