High frequency energy cascades in inviscid hydrodynamics

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ABSTRACT

With the aim of gaining insight into the notoriously difficult problem of energy and vorticity cascades in high dimensional incompressible flows, we take a simpler and very well understood low dimensional analog and approach it from a new perspective, using the Fourier transform. Specifically, we study, numerically and analytically, how kinetic energy moves from one scale to another in solutions of the hyperbolic or inviscid Burgers equation in one spatial dimension (1D). We restrict our attention to initial conditions which go to zero as $x \to \pm \infty$. The main result we report here is a Fourier analytic way of describing the cascade process. We find that the cascade proceeds by rapid growth of a crossover scale below which there is asymptotic power law decay of the magnitude of the Fourier transform.

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1. Introduction

It is sometimes useful to take a well understood problem and approach it from a new angle or viewpoint. The insight gained from the new perspective may help to allow further progress on more difficult problems. There are a number of very difficult open problems in the fields of hydrodynamics and mathematical fluid dynamics [1–5]. One of the most important unsolved problems concerns whether or not the three dimensional (3D) Euler equations for incompressible fluids admit finite time singularities, i.e. given an initially smooth velocity field, can the solution become singular or “blow up” in finite time? This problem is considered by experts to be very hard [3]. Slightly easier is the corresponding problem for viscous fluids. For Newtonian fluids, i.e. those that obey Fick’s law, the 3D Navier–Stokes equations play the role of the Euler equations. The question of whether or not initially smooth solutions of the 3D Navier–Stokes equations for incompressible fluids can blow up in finite time is one of the main open problems in mathematical fluid dynamics. A key aspect of both these blow-up problems is that in 3D, the kinetic energy which is present at given spatial scales (i.e., wave vector scales) can, in principle, shift to the next finer spatial scale (i.e., to smaller wave vectors). Meanwhile, the corresponding time scales also get smaller, so that the process could, in principle, be repeated at ever finer scales and at ever faster rates, resulting in a finite time singularity. This iterative downscaling of the kinetic energy is often described as a “cascade” and is what makes turbulence possible. For comparison, in 2D such kinetic energy cascades to ever smaller scales cannot lead to finite time singularities. In fact, there is a reverse cascade, where the energy at small scales is transferred to larger scales. But in 3D there is still no known mechanism which prohibits finite time blow-ups, because there is nothing to prevent the kinetic energy from cascading to ever smaller spatial and temporal frequency scales. Our goal here is to gain a better understanding of energy

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cascades from lower to higher frequencies by studying a simpler system: the 1D inviscid (or hyperbolic) Burgers equation [6, 7]. Here we use the 1D inviscid Burgers equation as a toy model to study energy cascades. The premise of this paper is that we can gain a deeper understanding of kinetic energy cascades in hydrodynamics by looking at the well understood inviscid Burgers equation.

In addition to the purely academic interest in this class of problems, there are some practical reasons for their study and we briefly mention a couple [8, 9]. The circulation of blood through the large blood vessels such as the Aorta and the Vena Cava can sometimes become turbulent instead of laminar. Turbulent flow is only possible if kinetic energy is able to cascade down to the smallest scales. A second example is turbulence in natural gas pipelines. Problems of a slightly different kind also arise in situations triggered by fluid instabilities. In these phenomena, in general, the onset of the instability is characterized by low frequency modes. The system then develops energy cascades and evolves rapidly to a turbulent regime described by high frequency oscillations and fluctuations at all scales. An example is given by the Rayleigh–Taylor instability, in which a dense inviscid fluid is supported by another fluid of lower density in a gravitational field. This instability occurs also in magneto-hydrodynamics and is responsible for the phenomenon called “Spread F” in the F-region of the ionosphere. Another situation, common in the petroleum industry, is generated when a less viscous fluid is injected in a petroleum reservoir, for oil recovery, at high injection rates. This can result in bubbles, vortices and also fluctuations at many scales. In addition, in porous media (such as petroleum reservoirs), even laminar flows could theoretically induce vortices of many sizes. For instance, there is some interest in studying fluid flows through rough walls and porous materials that form a fractal structure with sharp edges and open irregular cavities.

In the sections that follow, we briefly review the nonlinear wave equations of hydrodynamics and the properties of the inviscid Burgers equation. We then turn our attention to the definition of the $L^p$ norms, as well as the known conservation laws for the inviscid Burgers equation. We then report our numerical and analytical results, the main one being that the Fourier transform of initially smooth rapidly decaying solutions lies in $L^p$ for all $p > 1$. Finally, we present our results, discussion, and conclusions.

2. Burgers equation and hydrodynamics

The 3D Euler equations for incompressible fluids for the velocity field $u(x, t)$ are the following:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0$$

$$\nabla \cdot u = 0.$$  (1)

Here $p$ is the pressure. By taking the divergence of the first equation, one can invert the resulting elliptic equation to obtain $p = -\Delta^{-1} \text{Tr} (\nabla u)^2$, where $\Delta^{-1}$ is the integral operator which is the inverse of the Laplacian operator $\Delta = \nabla^2$. The Navier–Stokes equations differ from the Euler equation only by an additional term for viscous dissipation. The viscosity can be taken to be unity, by a simple renormalization of the units. Then, the Navier–Stokes equations become

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \Delta u.$$  (2)

The 1D viscous Burgers equation bears a resemblance to the Navier–Stokes equation (both are parabolic):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \Delta u.$$  (3)

Note that the viscous Burgers equation does not contain a pressure term. Indeed, Burgers equation represents a “fluid” which is perfectly compressible. The inviscid Burgers equation is simply

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$  (4)

i.e., there is no viscous dissipation. Like the Euler equations, the inviscid Burgers equation is hyperbolic and it is perhaps the simplest of the nonlinear hyperbolic wave equations.

At one time, it was conjectured that the viscous Burgers equation might be a good model of turbulence. It was hoped that the study of Burgers equation would lead to insights into the Navier–Stokes equations. However, a seminal result due independently to Hopf [10] and Cole [11] dashed those hopes. The Cole–Hopf transformation reduces the viscous Burgers equation to the heat equation, also known as the diffusion equation to physicists. The Cole–Hopf transformation is

$$u = -\frac{2}{\phi} \frac{\partial \phi}{\partial x} = -2 \frac{\partial}{\partial x} \ln \phi$$  (5)

allowing a closed form solution of the initial value problem given $u(x, 0) = u_0(x)$:

$$u(x, t) = -2 \frac{\partial}{\partial x} \ln \left[ \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} - \frac{1}{2} \int_{0}^{t} u_0(y) dy \right].$$  (6)
What this means is that the 1D viscous Burgers equation does not admit finite time singularities for initially smooth solutions. Nothing remotely similar to turbulence is possible. In slightly different language, the kinetic energy cannot cascade to ever higher spatial and temporal frequencies and blow-ups are ruled out.

However, the inviscid Burgers equation has a very different behavior: finite time singularities are guaranteed for any (non-trivial) initially smooth solution which decays to zero at positive and negative infinity (e.g., Schwartz functions). One reason for this big difference between the viscous and inviscid Burgers equation is that the viscous dissipation term is a singular perturbation, i.e., it contains a differential operator of order higher than those in the original equation.

In the rest of this section, we review the relevant points concerning Burgers equation, all of which are well known. We refer the interested reader to Refs. [7,12–14].

2.1. Solutions of the inviscid Burgers equation

2.1.1. Separation of variables

We briefly review the main properties of solutions, starting with separable solutions which can be expressed as \( u(x, t) = f(x)g(t) \). Then,

\[
\frac{\partial}{\partial t} (fg) = -fg \frac{\partial}{\partial x} (fg) = -fg^2 \frac{d}{dx} f
\]

from which we get

\[
- \frac{1}{g^2} \cdot \frac{d}{dt} g = \frac{d}{dx} f = \text{constant}.
\]

The general solution is thus

\[
u = \frac{x_0 - x}{t_c - t},
\]

where \( x_0 \) and \( t_c \) are constants of integration. The solution is zero and stationary at \( x = x_0 \) and the solution blows up at the critical time \( t = t_c \), when the solution becomes discontinuous. We also note that this equation also solves the parabolic or viscous Burgers equation (4).

Most solutions of the inviscid Burgers equation are not separable. However, one could approximate smooth functions by piecewise linear solutions which are piecewise separable. We do not take this approach. Instead, we will use (10) to obtain an exact solution for the Fourier transform of \( u \), as a way to get some intuition about kinetic energy cascades.

2.1.2. Implicit solutions

The inviscid Burgers equation (5) has a very simple physical interpretation. The solution \( u(x, t) \) can be thought of as the velocity of a free particle at the point \( x \) at time \( t \). So at a subsequent time \( t + dt \), the particle will be at position \( x + u dt \).

Indeed, given the initial condition \( u_0(x) = u(x, 0) \), the solution satisfies

\[
u(x + tu(x, t), t) = u_0(x).
\]

The method of characteristics, for example, can be used to obtain this relation [7]. We will make heavy use of this implicit solution for numerically integrating the inviscid Burgers equation.

Indeed, the usual numerical recipes for integration, such as Runge–Kutta, give relatively bad results because of the extreme instability of the inviscid Burgers equation, which, like the Euler equation, is a hyperbolic wave equation. The nonlinear term causes errors to grow very rapidly, in an uncontrolled manner. It is, of course, possible to smooth out the errors by adding a perturbation, but then one in fact has a different partial differential equation. Our interest is in the actual inviscid Burgers equation, not in smoothed out perturbations of it. For our purposes, Runge–Kutta and similar approaches are not helpful. Instead, using (11) as a “trick” we can completely avoid the hyperbolic numerical instability.

2.2. Conservation laws for \( L^p \) norms

2.2.1. The Lebesgue \( p \)-norms

The \( L^p \) norms are defined for \( 1 \leq p < \infty \) in terms of the Lebesgue integral of the \( p \)th powers of the absolute value of a given function \( f \) on the reals:

\[
\|f\|_p := \left( \int_{\mathbb{R}} |f|^p \, dx \right)^{\frac{1}{p}}.
\]

The \( L^\infty \) norm, which is the limit of the \( L^p \) norm as \( p \to \infty \), is the essential supremum norm (i.e. the supremum norm which takes into account the Lebesgue philosophy of “almost everywhere” equivalence). For our purposes, the Riemann integral is good enough.
The 1D inviscid Burgers equation has a conservation law of $L^p$ norms, at least for sufficiently regular solutions:

$$\frac{d}{dt} \|u\|_p^p = \int_{\mathbb{R}} \frac{\partial}{\partial t} u^p \, dx$$

(13)

$$= \int_{\mathbb{R}} p u^{p-1} \frac{\partial}{\partial t} u \, dx$$

(14)

$$= -p \int_{\mathbb{R}} u^p \frac{\partial}{\partial x} u \, dx$$

(15)

$$= -\frac{p}{p+1} \int_{\mathbb{R}} \frac{\partial}{\partial x} u^{p+1} \, dx$$

(16)

$$= 0.$$  

(17)

The last integral is zero because $u$ is assumed to decay to zero as $x \to \pm \infty$. In particular, the $L^1$ and $L^2$ norms are conserved.

This calculation is valid for $u \geq 0$ but can be extended to real $u$ under some natural conditions, given that the points $x$ where $u(x, t) = 0$ are stationary. We are also assuming that the solution is differentiable, but the result can be extended to less regular solutions with some care.

Most importantly for our purposes, the $L^\infty$ norm is conserved. The proof follows directly from (11).

2.3. $L^p$ norms of the Fourier transform

We now focus on the Fourier transform on the space variable $x$. Recall that the Fourier transform $\hat{f}$ of a function $f$ and the inverse transform are such that the Fourier transform of the Fourier transform is the original function except for a minus sign. There are several conventions which differ in where the factors $\sqrt{2\pi}$ appear. Without loss of generality, we will use the following definitions:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx$$

(18)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} \, dk.$$  

(19)

By the theorems of Parseval and Plancherel, the $L^2$ norm of the Fourier transform $\hat{u}$ of $u$ is also conserved. In Banach space theory, it is well known that for all $1 \leq p < \infty$ (but not $p = \infty$) the dual of $L^p$ is $L^q$, with

$$\frac{1}{p} + \frac{1}{q} = 1, \quad q > 1.$$  

(20)

In particular, $L^2$ is self-dual. The dual of $L^\infty$ is a special case which we will not discuss here, because $L^\infty$ is not a separable Banach space. The dual of $L^1$, however, is $L^\infty$.

So one might be tempted to think that since the $L^\infty$ norm of $u$ is conserved, therefore the $L^1$ norm of the Fourier transform might also be conserved. This is not true. The analog of Plancherel's theorem for $p \neq 2$ is the Hausdorff–Young inequality. Since it is an inequality rather than an equality, there is no conservation law. In principle, not only is the $L^1$ norm of $\hat{u}$ not conserved, it need not even be finite, unless there is some other additional reason for finiteness.

2.4. The Fourier transform and energy cascades

Our main goal here will be to study the evolution of the Fourier transform $\hat{u}(k, t)$. We will see below that the $L^p$ conservation laws allow some insight into the cascade process. The energy distribution is given by the modulus squared of the Fourier transform. So, if a solution breaks down at time $t = t_c$, then the absolute value of the Fourier transform at the time of break down must decay as a power law (asymptotically). This is because the (inverse) transform of a rapidly decaying function is smooth. So long as the Fourier transform is rapidly decaying, the solution cannot break down. Specifically, the spatial derivative $\partial^n u/\partial x^n$ of order $n$ is bounded by the $n$-th moment of the absolute value of the Fourier transform. (These are well known properties of the Fourier transform and can be found in any book on real or Fourier analysis.)

Statistical moments of a distribution are always finite unless the distribution has an asymptotic power law decay. The cascade process must therefore shift the energy from low frequencies to high frequencies in a manner such that at the
Fig. 1. (a) A sinusoidal initial condition and subsequent time evolution. The dashed lines show how the solutions would look like if continued past the time of first breakdown, when the solution becomes multi-valued. (c) Truncated Gaussian initial condition and (e) symmetric exponential initial condition of the form $u_0 = \exp[-|x|]$. (b), (d) and (f) show the double log plots of the magnitude of the Fourier transform, at various times for the solutions shown in (a), (c), and (e). Since straight lines on double log plots indicate power laws, we see that a power law tail forms as the time approaches the blow-up time $t_c$. (In fact, unless the power law tail forms, the solution must remain smooth, due to basic theorems of Fourier analysis.) We thus see that the cascade process forms the finite time singularity by sprouting a power law tail.

At time $t_c$ of first breakdown, the Fourier transform $\hat{u}$ acquires an asymptotic power law tail. It is this process that we wish to understand qualitatively and quantitatively. Our goal is to study in great detail how and why this power law tail forms.

Fig. 1(a) shows a sinusoidal initial condition and the solution at subsequent times. This solution is not rapidly decaying, but we can instead use periodic boundary conditions to avoid the “problems at infinity”. At times beyond the time of first breakdown, the solution is no longer a function, because it becomes multi-valued. The solution has been numerically integrated using the previously mentioned trick, bypassing the Runge–Kutta and similar integration algorithms which are problematic for hyperbolic partial differential equations. Fig. 1(b) shows the absolute value of the Fourier transform on a double log scale. We see that as the solution gets closer to the time $t_c$, the Fourier transform acquires a power law tail $\sim k^{-1}$.

Fig. 1(c) shows a Gaussian initial condition and subsequent evolution. Fig. 1(d) shows the absolute value of the Fourier transform at subsequent times. Again, we see similar behavior as with the sinusoidal initial condition.

Fig. 1(e) shows an initial condition which is already singular, and the subsequent evolution. Fig. 1(f) shows the absolute value of the Fourier transform at subsequent times. Note that the Fourier transform is not rapidly decaying even for the initial condition, i.e. there is asymptotic decay even before the time $t_c$. However, the power law exponent clearly changes. As we approach the time $t_c$, the power law tail becomes “fatter”.
The main finding we see here is that at time \( t_c \) the absolute value of the Fourier transform decays as \( \sim k^\alpha \), with \( \alpha \) never smaller than 1. The cascade process is thus incapable of generating power law tails which are fatter than \( \sim 1/k \).

3. Results

3.1. Closed form expression for a special case

We are unable to obtain closed form expressions when we integrate analytically the above initial conditions. In order to make progress analytically, we consider a triangular initial condition

\[
    u_0(x) = \begin{cases} 
        0, & |x| > 1 \\
        x + 1, & -1 \leq x \leq 0 \\
        1 - x, & 0 < x \leq 1.
    \end{cases}
\]  

(21)

This initial condition is not smooth. However, there are precisely three singular points, two of which are stationary in time. Everywhere else, the initial condition is smooth. See Fig. 2(a).

At subsequent times \( 0 < t < 1 \), the solution is

\[
    u(x, t) = \begin{cases} 
        0, & |x| > 1 \\
        \frac{x + 1}{1 + t}, & -1 \leq x \leq t \\
        \frac{1 - x}{1 - t}, & t < x \leq 1.
    \end{cases}
\]  

(22)

The solution clearly breaks down at time \( t_c = 1 \). See Fig. 2(a).
The Fourier transform \( \hat{u} \) (up to breakdown) is

\[
\hat{u}(k, t) = \frac{\int_{-1}^{1} e^{ikx} \left( \frac{x+1}{t+1} \right) \, dx + \int_{-1}^{1} e^{ikx} \left( \frac{x-1}{t-1} \right) \, dx}{\sqrt{2\pi}}
\]

\[
= \frac{e^{-ik} + e^{ik} - 2e^{ikt} - t(e^{-ik} - e^{ik})}{\sqrt{2\pi} (t^2 - 1)k^2}.
\]

Taking the absolute value, we get

\[
|\hat{u}(k, t)| = \frac{|(1-t) + (1+t)e^{ikt} - 2e^{ikt(1+t)}|}{\sqrt{2\pi} |t^2 - 1| k^2}.
\]

Note that at time \( t = 0 \) the asymptotic magnitude decay becomes

\[
|\hat{u}(k, 0)| \sim k^{-2}.
\]

In fact, this is the correct decay for any \( 0 \leq t < 1 \).

However, at the time \( t = t_c = 1 \) the magnitude becomes

\[
|\hat{u}(k, t_c)| = \frac{\sqrt{1 + 2k^2 - 2\cos(2k) - 4k\cos(k)\sin(k)}}{2k^2\sqrt{\pi}}
\]

\[
\sim k^{-1}.
\]

This exact solution thus allows us to see exactly how the cascade proceeds. At initial time \( t = 0 \), the Fourier transform has inverse square decay. Note that it is not rapidly decaying, due to the three singular points in the triangular initial condition. However, as time proceeds, the inverse square power law \( 1/k^2 \) gives way to an inverse power law \( 1/k \). When the solution breaks down at time \( t = t_c \), the \( L^1 \) norm of the Fourier transform \( \hat{u} \) becomes infinite (because the integral of \( 1/k \), which is \( \ln k \), diverges).

How does the power law \( 1/k^2 \) change over time to the power law \( 1/k \)? Intuitively, one expects that the cascade process shifts the energy to ever higher frequencies, so that a crossover (e.g., a “knee joint”) forms between the two scaling regimes separating the \( 1/k^2 \) behavior from the \( 1/k \) behavior, with the latter regime eventually completely substituting the former regime and extending all the way to \( k = \pm\infty \) at time \( t_c \). We expect the scale of the crossover to get larger and larger as \( t \) grows, becoming infinite at time \( t = t_c \).

Indeed, this intuition is correct, as seen in Figs. 2(b) and 3. The last figure shows a plot of the Fourier transform \( \hat{u} \) at various times. The shift of the crossover scale is clearly visible.

Since (25) is an exact solution (which we were fortunate enough to find), we need not worry that this finding is a numerical artifact or error, etc. Indeed, the fact that the solution is exact lends strong credibility to the results shown in Fig. 3. Notice how the \( 1/k \) scaling seen in the previous numerical results has reappeared in the exact asymptotic scaling shown in (28).

We briefly comment on what happens after the time \( t_c \) of first breakdown. The solution (22), when continued to \( t > 1 \) or to \( t < -1 \) becomes multi-valued, so it ceases to be a function. Even though there is no longer a classical or strong solution, it is possible to think in terms of suitably defined weak solutions. Usually, weak solutions are defined in terms of distributions, but here we use the term “weak” to mean, loosely speaking, a generalized function of some kind. The dashed lines in Fig. 2(a) show the (weak) solution at times when it is no longer a (single valued) function. However, we can relax the requirement that the solution be a function and study the solution as a curve.

### 3.2. Fourier transform for the general case

The problem of establishing a general power law result depends on our handling of the Fourier magnitude. So far, this has been accomplished graphically (in Section 2.4) or using closed form solutions (Section 3.1, see (25) and (27)). In what follows, we suggest a more general treatment of the Fourier transform of the Burgers solution.

Let us define a new variable \( y = x - ut \) and change variables to put everything in terms of \( y \). Let us also use the notation \( \partial_x \) or \( \partial_{\nu} \) for partial differentiation, for conciseness. Then, using (11), we have \( u(x, t) = u(y + ut, t) = u_0(y) \). Moreover,

\[
\partial_y y = (1 - t \partial_x u(x, t))
\]

\[
\partial_y u(x, t) = (\partial_x u_0(y))(\partial_x y)
\]

\[
= (\partial_x u_0(y))(1 - t \partial_x u(x, t))
\]

\[
\partial_y u_0(y) = (\partial_x u(x, t))(1 + t \partial_x u_0(y))
\]

\[
\partial_x u(x, t) = \frac{\partial_y u_0(y)}{1 + t \partial_x u_0(y)}.
\]
Fig. 3. Magnitude of the Fourier transform $|\hat{u}(k)|$ versus $k$ on double log scale for various times $t$ for the solution shown in the previous figure and discussed in the text. The crossover scale between the $1/k^2$ and $1/k$ scaling regimes gets larger as the time approaches the blow-up time $t_c$. The growth of the $1/k$ “fat tail” shows very clearly how the cascade happens. These figures have been made using the exact solution (see Eq. (25)) and so we can discard numerical error or artifacts. The progressive shift of the crossover scale as time increases shows, dramatically, the mechanism by which the cascade process is able to form singularities in finite time.

We can now calculate the time-dependent Fourier transform:

$$\hat{u}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ikx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(y) e^{-ik(y+u(x,t)t)} \, dy$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1-t\partial_x u(x,t)} \, dy$$
Acknowledgments

L. known conservation laws for the time and the Fourier transform

\[ \frac{\partial}{\partial t} \theta + \mathbf{u} \cdot \nabla \theta = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -(-\nabla^2)^{-1/2} \nabla \cdot \mathbf{u} \theta. \]

Here \( \nabla^\perp \) is a differential operator which satisfies \( \nabla^\perp \cdot \nabla^\perp = 0 \), and \( \theta \) is the scalar temperature. Numerical simulations seem to suggest that there are no finite time singularities. Indeed, the cascade of kinetic energy is much less violent than for the inviscid Burgers equation because the incompressibility condition \( \nabla \cdot \mathbf{u} = 0 \) forces the Fourier transform \( \hat{\mathbf{u}} \) to have no radial component. However, to date, no reason is known why the maximal vorticity \( \| \nabla^\perp \theta \|_{L^\infty} \) should not, in principle, blow up in finite time. The known conservative norms of \( \theta \) and its derivatives (e.g., Sobolev norms) do not prohibit blow-up.

In summary, we have studied the asymptotic decay of the Fourier transform of solutions at the time of breakdown. Although the kinetic energy, which is the square of the \( L^2 \) norm, is conserved, the distribution of the energy evolves in time and the Fourier transform \( \hat{\mathbf{u}} \) acquires a power law tail at the time of first breakdown—a consequence of the cascade of energy to ever higher spatial frequencies. Thus far, the only way to understand what is happening is by reference to the known conservation laws for the \( L^p \) norms of the solutions.

4. Discussion and conclusion

The main new result reported here is the crossover seen in Fig. 3. We have found that when solutions of the inviscid Burgers equation break down, the resulting singularity has a well defined scale invariance symmetry. Specifically, the Fourier transform decays as a power law and for initially smooth solutions the decay is never slower than \( \sim 1/k \).

Why should this be? The \( L^p \) norm of both \( u \) and \( \hat{u} \) is also conserved, which means that \( \hat{u} \) must decay at least as rapidly as \( \sim k^{-1/2} \). But the observed decay of \( \sim 1/k \) at the time of breakdown is much faster than the decay \( \sim k^{-1/2} \) expected from energy (i.e. \( L^2 \)) considerations.

Let \( \hat{u}_0(x) = u(x, t_c) \) be the solution at the time of first breakdown. Since \( \hat{u}_0 \) has a well defined Fourier transform \( \hat{\hat{u}}_0 \), it must be possible to obtain \( \hat{u}_0 \) as the inverse transform of \( \hat{\hat{u}}_0 \). However, one cannot define Fourier transforms or their inverses in a well defined manner for non-\( L^2 \) functions. Instead, one must take limits. For example, the Heaviside step function has a well defined Fourier transform, but we cannot take the inverse transform to obtain the Heaviside step function without taking the limit, because the Fourier transform of the Heaviside step function does not lie in \( L^1 \).

We can now answer why \( |\hat{u}| \) decays no slower than \( \sim 1/k \). Recall that \( u \) lies in \( L^p \) for \( 1 \leq p < \infty \). So the Fourier transform \( \hat{u} \) lies in \( L^q \) for \( 1 \leq q < \infty \). If \( \hat{u} \) decayed as \( \sim k^{-1+\epsilon} \) with \( \epsilon > 0 \) then \( \hat{u} \) could not lie in \( L^q \) for \( q \leq 1/(1+\epsilon) \), leading to contradiction. (In fact, one might try to extend this result to \( q = 1 \) by trading some regularity, perhaps by using weak \( L^p \) or Lorentz norms.) We can now see why the absolute value of the Fourier transform has an asymptotic power law tail which is bounded by \( 1/k \). The conservation law of \( L^p \) norms controls the “fat tails” of the Fourier transform at the time of first breakdown. The decay cannot be “fatter” than \( 1/k \).

We conclude with a discussion of the broader context. The cascade we have studied here might be relevant to other hyperbolic wave equations. For example, in addition to the inviscid Burgers equation and the Euler equations, another hyperbolic equation is the 1D Constantin–Lax–Majda [15] model which was proposed as a “toy model” for the 3D Euler equations. Yet another hyperbolic system is the 2D quasi-geostrophic equations [18,19,16,17]. It is known that the 1D CLM model can produce finite time singularities; as to the 2D quasi-geostrophic model, the issue is still undecided. The 2D quasi-geostrophic equations can be expressed as follows:

\[ \frac{\partial}{\partial t} \theta + \mathbf{u} \cdot \nabla \theta = 0, \]

\[ \mathbf{u} = -(-\nabla^2)^{-1/2} \nabla^\perp \theta. \]

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References