Superdiffusion driven by exponentially decaying memory

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Abstract. A superdiffusive random walk model with exponentially decaying memory is reported. This seems to be a self-contradictory statement, since it is well known that random walks with exponentially decaying temporal correlations can be approximated arbitrarily well by Markov processes and that central limit theorems prohibit superdiffusion for Markovian walks with finite variance of step sizes. The solution to the apparent paradox is that the model is genuinely non-Markovian, due to a time-dependent decay constant associated with the exponential behavior.

Keywords: phase transformations (theory), stochastic processes (theory), diffusion
1. Introduction

Diffusion processes and random walks have been extensively used to describe important phenomena in many areas, such as physics, chemistry and biology [1]. The random walk and its generalization, the continuous time random walk model introduced by Montroll and Weiss in 1965 [2], are important tools for the study of many physical phenomena, such as in disordered media [3]–[6], earthquake modeling [7] and financial markets [8].

A basic fact which physicists learn early in their careers is that exponentially decaying correlations cannot lead to long-range order. For example, the Ising model with nearest neighbor interactions in one dimension cannot sustain long-range order at nonzero temperatures [9]. For the same reason, random walks with exponentially decaying correlations and whose step sizes have finite variance behave similarly to standard uncorrelated Brownian random walks at long times, with the mean squared displacement scaling linearly with time. Moreover, any random walk model with exponentially decaying memory can be modeled as an $n$-step Markov process, i.e. a Markov process in which the current transition probability depends only on the previous $n$ steps taken. No matter how large we choose $n$, at long times the mean squared displacement necessarily scales linearly in time because of the central limit theorem. Specifically, at large times the memory becomes negligible so that upon renormalizing, i.e. coarse-graining, one recovers the uncorrelated Brownian random walk as a fixed point attractor of the renormalization flow map, so anomalous diffusion [10]–[19] is not possible. These are well known facts. It was thus a surprise to us when we found an apparent counter-example. We report here a random walk model with exponentially decaying memory which is superdiffusive even at long times, i.e. the mean squared displacement grows superlinearly in time, rather than linearly.

The resolution of the (apparent) paradox reveals a gap in how the subject is usually considered. Indeed, we show by construction that it is in fact possible to have a genuinely non-Markovian random walk model with exponentially decaying memory, provided that the decay constant is time-dependent.
2. Model

The model we study is a variant of the so-called elephant random walk (ERW) model proposed by Schütz and Trimper [20]. The random walker keeps a record of the entire history of the walk, so that the walk is non-Markovian in principle. Many variants of this model have been proposed, such as the ‘Alzheimer walk’ model which led to the unexpected findings of amnestically induced superdiffusion and log-periodic superdiffusion (e.g., see [14]). Here, we propose a model with an exponentially decaying memory profile. This model is inspired by another recently proposed model [21] which had a (truncated) Gaussian memory profile. In this work we essentially replace the Gaussian by an exponential.

The ERW model, using the notation introduced in [20], starts at the origin at time $t_0 = 0$ and retains memory of its complete history. In each time step the walker moves one step to either the right or the left, i.e.,

$$x_{t+1} = x_t + v_{t+1}$$  \hfill (1)

where $v_{t+1}$ represents a stochastic noise with two-point autocorrelations (i.e. memory). The walker can remember the entire history of prior random walk step directions $\{v_t\}$ for $t' \leq t$. At time $t$, one randomly chooses a random time $1 \leq t' \leq t$ with equal a priori probabilities. The current step direction $v_t$ is then chosen based on the value of $v_{t'}$ as

$$v_{t+1} = \begin{cases} 
  +v_{t'}, & \text{with probability } p \\
  -v_{t'}, & \text{with probability } 1 - p.
\end{cases}$$  \hfill (2)

Without loss of generality, it is assumed that the first step always goes to the right, i.e. $v_1 = +1$. The position at time $t$ thus follows

$$x_t = \sum_{t'=1}^{t} v_{t'}$$  \hfill (3)

and the second moment is given by

$$\langle x_t^2 \rangle = \begin{cases} 
  \frac{t}{3 - 4p}, & p < 3/4 \\
  t \ln t, & p = 3/4 \\
  t^{4p-2}/[\Gamma(4p-3)\Gamma(4p-2)], & p > 3/4
\end{cases}$$  \hfill (4)

which are exact relations valid in the asymptotic limit. The ERW presents a superdiffusive regime ($p > 3/4$) and a localized regime ($p < 3/4$), with $p = 3/4$ being marginally superdiffusive. Interestingly, for $1/2 < p < 3/4$, the square of the mean does diverge, but more slowly than the mean square displacement, so that the behavior remains diffusive. This regime is termed an escape regime with a mean displacement given by $\langle x_t \rangle \sim t^{2p-1}$.

The exact propagator is reported to be a Gaussian distribution [20], i.e.,

$$P(x, t) = \frac{1}{\sqrt{4\pi D(t)}} \exp \left( -\frac{(x - \langle x(t) \rangle)^2}{4tD(t)} \right)$$  \hfill (5)

where $D(t, p) = (1/8p - 6)[(t/t_0)^{4p-3} - 1]$ is the time- and $p$-dependent diffusive coefficient. Within the superdiffusive regime the distribution has been found to be non-Gaussian [24].

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In this paper, we focus on a random walker with the ability to recall previous events, but with more recent events remembered more frequently—or easily—than those from the more distant past. We shall refer to this model as the exponential memory model. While in the ERW model the previous time \( t' \) is chosen from a uniform distribution, in the exponential memory model \( t' \) is randomly chosen from an exponential probability distribution. The probability of choosing a previous time \( t' \) is then given by

\[
P_\lambda(t', t) = A \exp \left( -\frac{\lambda(t - t')}{t} \right),
\]

where \( A \) is a normalization constant. The parameter \( \lambda \) adjusts the shape of the exponential distribution in the usual manner, but unlike typical decay constants \( \lambda \) is adimensional.

Unfortunately, this model does not yet have a known exact solution. Nevertheless, an approximate solution can be found by assuming that the exponential memory pattern can be mapped onto an equivalent memory profile with a—smaller—rectangular window size. Within our approach, this basic ansatz is necessary to get an approximate exact solution to the non-Markovian exponential memory model. Its validity is supported by the numerical results shown below. A non-Markovian rectangular memory profile model has a fixed memory size \( L = ft \), where \( 0 < f < 1 \) is a new parameter that fixes the size of the memory. The probability of choosing a previous \( t' \) is given simply by \( 1/L \) for \( (1 - f)t < t' \leq t \), and zero otherwise. This memory profile is flat, or constant, and has the shape of a rectangle instead of an exponential, such that the more ancient memories, i.e., those that occurred prior to time \((1 - f)t\), are forgotten. The random walker can therefore recall only a fraction \( f \) of the more recent steps.

3. Results

The main idea now is to determine an effective fraction \( f_{\text{eff}}(\lambda) \) which makes the model with a rectangular memory pattern with \( f = f_{\text{eff}} \) behave just the same as the exponential memory model with a given \( \lambda \). Then the Fokker–Planck equation should be equivalent for both models, following the ideas discussed in [23]. We can define the memory’s effective length for the exponential memory model by

\[
L \equiv \int_0^t \left[ P_\lambda(t', t)/P_{\text{max}}(t', t) \right] \, dt'
\]

where \( P_{\text{max}}(t', t) \) is the maximum value of \( P_\lambda(t', t) \). Using (6) we get

\[
L = \int_0^t e^{-\lambda(t-t')/t} \, dt' = \left( \frac{1 - e^{-\lambda}}{\lambda} \right) t
\]

which gives, using \( L = f_{\text{eff}}t \),

\[
f_{\text{eff}} = (1 - e^{-\lambda})/\lambda.
\]

Equation (9) is the key result that allows us to perform the mapping between the two models and achieve an approximate solution for the exponential memory model by reducing it to a rectangular memory model. We shall refer to the model with a rectangular
memory profile with \( L = f_{\text{eff}} t \) as the simplified model. This model has already been studied before [22], and its solution can provide a good analytical solution for the exponential memory model. The analytical solution can be obtained by first defining \( n_{\text{f}}(t) \) and \( n_{\text{b}}(t) \) as the numbers of steps taken forward and backward, respectively, up to a given time \( t \). Therefore, the total number of steps taken forward within the time interval \([t - L, t]\) can be written as \( \Delta n_{\text{f}} = n_{\text{f}}(t) - n_{\text{f}}(t - L) \). Similarly, the total number of steps taken backward in the same time interval is written as \( \Delta n_{\text{b}} = n_{\text{b}}(t) - n_{\text{b}}(t - L) \). Thus, the effective probabilities of taking a step forward and backward, i.e., \( P_{\text{eff}}^{+}(t, x) \) and \( P_{\text{eff}}^{-}(t, x) \), respectively, for \( t > 0 \) are given by

\[
\begin{align*}
P_{\text{eff}}^{+}(t, x) &= (\Delta n_{\text{f}}/L)p + (\Delta n_{\text{b}}/L)(1-p) \\
P_{\text{eff}}^{-}(t, x) &= (\Delta n_{\text{b}}/L)p + (\Delta n_{\text{f}}/L)(1-p).
\end{align*}
\]

Taking the difference between equations (10) and (11), with \( n_{\text{f}}(t) + n_{\text{b}}(t) = t \) and \( n_{\text{f}}(t - L) + n_{\text{b}}(t - L) = t - L \), we obtain an expression for the effective, or expected, value of \( v \) at time \( t + 1 \), i.e.,

\[
v_{\text{eff}}^{t+1} = P_{\text{eff}}^{+}(t, x) - P_{\text{eff}}^{-}(t, x).
\]

Therefore, we can write \( \Delta n_{\text{f}} + \Delta n_{\text{b}} = L \) and \( x_{\text{f}} = n_{\text{f}}(t) - n_{\text{b}}(t) + x_{0} \), and also \( x_{(t-L)} = n_{\text{f}}(t - L) - n_{\text{b}}(t - L) + x_{0} \), which gives \( x_{\text{f}} - x_{(t-L)} = \Delta n_{\text{f}} - \Delta n_{\text{b}} \). Then we have \( \Delta n_{\text{f}} = [L + x_{\text{f}} - x_{(t-L)}]/2 \) and \( \Delta n_{\text{b}} = [L - (x_{\text{f}} - x_{(t-L)})]/2 \). We can thus rewrite equation (12) as

\[
v_{\text{eff}}^{t+1} = \alpha \frac{x_{\text{f}} - x_{(t-L)}}{L}
\]

where \( \alpha = 2p - 1 \).

The conditional probability that the walker is at the position \( x \) at time \( t + 1 \) given the earlier position \( x_{0} \) at \( t = 0 \) is given by

\[
P(x, t + 1|x_{0}, 0) = P(x + 1, t|x_{0}, 0)P^{-}(t, x + 1) + P(x - 1, t|x_{0}, 0)P^{+}(t, x - 1).
\]

Now using \( \Delta n_{\text{f}} + \Delta n_{\text{b}} = L \) and \( \Delta n_{\text{f}} - \Delta n_{\text{b}} = x_{\text{f}} - x_{(t-L)} = x - G(x) \) and also the definitions (10) and (11) again, we obtain

\[
\begin{align*}
P_{\text{eff}}^{+}(t, x) &= \frac{1}{2} \left[ 1 + \alpha \left( \frac{x - G(x)}{L} \right) \right] \\
P_{\text{eff}}^{-}(t, x) &= \frac{1}{2} \left[ 1 - \alpha \left( \frac{x - G(x)}{L} \right) \right].
\end{align*}
\]

Substitution of equations (15) and (16) into (14) gives

\[
P(x, t + 1|x_{0}, 0) = \frac{1}{2} \left[ 1 - \alpha \left( \frac{x + 1 - G(x + 1)}{L} \right) \right] P(x + 1, t|x_{0}, 0) \\
+ \frac{1}{2} \left[ 1 + \alpha \left( \frac{x - 1 - G(x - 1)}{L} \right) \right] P(x - 1, t|x_{0}, 0).
\]

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We now introduce the notation $P(x - x_0, t - t_0)$ for the propagator $P(x, t|x_0, t_0)$. By subtracting $P(x, t)$ from both sides of the expression above we obtain

$$P(x, t + 1) - P(x, t) = \frac{P(x + 1, t) - 2P(x, t) + P(x - 1, t)}{2} - \frac{\alpha}{L} \left[ \frac{[x + 1 - G(x + 1)] P(x + 1, t) - [x - 1 - G(x - 1)] P(x - 1, t)}{2} \right].$$

For large $t$ we can write $L = ft$ and $G(x) = x_{(1-f)t}$. Thus, in the continuum limit, taken in the usual manner, we can obtain an approximated FP equation for the propagator [24], i.e.,

$$\frac{\partial P(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2} - \frac{\alpha}{ft} \frac{\partial}{\partial x} \left[ xP(x, t) - x_{(1-f)t}P(x, t) \right].$$

The displacement $x_{(1-f)t}$ in equation (17) can be correlated with the displacement $x = x_t$ by writing $x_{(1-f)t} = xh_{1-f}(x, t)$, which defines the stochastic function $h_{1-f}(x, t)$. Notice that this function can assume non-positive values. Using this definition, we can write the mean value of $x_{(1-f)t}$ as

$$\langle x_{(1-f)t} \rangle = \int_{-\infty}^{+\infty} xh_{1-f}(x, t)P(x, t) \, dx$$

and since by definition the mean value of $x_{(1-f)t}$ is

$$\langle x_{(1-f)t} \rangle = \int_{-\infty}^{+\infty} xP(x, (1-f)t) \, dx$$

we can write

$$h_{1-f}(x, t) = \frac{P(x, [1-f]t)}{P(x, t)} + \frac{g_{1-f}(x, [1-f]t)}{P(x, t)}.$$  \hspace{1cm} (20)

Here, $g_{1-f}[x, (1-f)t]$ is another stochastic function which must satisfy $\int_{-\infty}^{+\infty} xg_{1-f}(x, (1-f)t) \, dx = 0$. For $f = 1$ we obtain

$$h_0(x, t) = \frac{P(x, 0)}{P(x, t)} + \frac{g_0(x, 0)}{P(x, t)}$$  \hspace{1cm} (21)

where, for $x \neq 0$, $P(x, 0) = \delta_{x, 0} = 0$ and $g_0(x, t) = \delta_{x, 0} = 0$. This is the condition for $h_0 = 0$, such that equations (18) and (19) satisfy the initial condition $x_0 = 0$. Now using equations (17) and (20), we can write

$$\frac{\partial P(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2} - \frac{\alpha}{ft} \frac{\partial}{\partial x} \{ xP(x, t) - x[P(x, [1-f]t) + g_{1-f}(x, [1-f]t)] \}. $$

For $\lambda \to 0$, one obtains $f = 1$, which leads to the FP equation of Schütz and Trimper [20].

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Figure 1. The Hurst exponent $H$ as a function of the parameter $p$ for the long-range memory correlated model with exponentially decaying memory. The symbols represent numerically evaluated values for the Hurst exponent for several values of $\lambda$, which represents a measure of the memory length. Cubic splines were used to draw the $H$ versus $p$ lines. Known analytic results for the elephant random walk (ERW) (longest possible memory model) and the traditional Brownian random walk (RW), which has no memory, representing extreme cases of memory length, are also shown for comparison as dashed blue and red lines, respectively. The inset shows a zoomed area of the $H > 1/2$ portion of the main curve. We notice a transition from normal diffusion ($H = 1/2$) to superdiffusion ($H > 1/2$) for $p = 3/4$ (exact result for the ERW model). We clearly see that the exponential memory model presents superdiffusion ($H > 1/2$) for a range of values of $\lambda$, contradicting the common belief that exponential memory cannot give rise to superdiffusion.

4. Discussion and conclusion

The main point to note in the above analytical results is the non-Markovian nature of the random walk. For any finite $\lambda$, the memory window $L = f_{\text{eff}} t$ grows linearly in time (and similarly for $f t$ in the simplified model). In this technical sense, the memory is scale-free. It is the absence of a characteristic time scale that makes it impossible to renormalize these models to recover the usual Brownian random walk for all $\lambda$ or $f$.

We now discuss the numerical results. The Hurst exponent $H$ for a random walk with drift is defined by the scaling relation $\langle (x - \langle x \rangle)^2 \rangle \sim t^{2H}$ at suitably long times. Here, we use the simpler definition $\langle x^2 \rangle \sim t^{2H}$, which is valid as long as the mean grows in time less quickly than the standard deviation. In the figures that follow, the averages are calculated over $10^4$ walks of length $10^7$.

Figure 1 shows the behavior of the Hurst exponent $H$ as a function of the parameter $p$ for some values of $\lambda$ for the exponential memory model. We compare the extreme cases of this model with the known analytical results for the ERW model and the results for the traditional random walk (RW). From the simulation results we observe the existence of a transition from normal diffusion ($H = 1/2$) to superdiffusion ($H > 1/2$) for $p = 3/4$ (exact solution of the ERW model). The inset emphasizes the superdiffusive region. We clearly see superdiffusion for the random walker with exponentially decaying memory.
Figure 2. (a) Plot of the Hurst exponent $H$ as a function of both the parameter $p$ and $e^{-\lambda}$ for several values of $\lambda$, for the exponential memory model. (b) The same as (a) but for the equivalent rectangular memory model. For large values of $\lambda$ ($\lambda \to \infty$) the memory length becomes very small and the model behaves like the traditional random walk (RW). For $\lambda$ small ($\lambda \to 0$), the memory length tends to the full memory profile, recovering the elephant random walk (ERW) model. Notice how the exponential memory model and the simplified model have essentially the same behavior, justifying the mapping ansatz that we use to connect the two models. Notice also that the two models allow superdiffusion.

Figure 2 shows the Hurst exponent $H$ as a function of both the parameter $p$ and $e^{-\lambda}$, for several values of $\lambda$. Figure 2(a) displays the behavior of the exponential memory model, whereas figure 2(b) exhibits the behavior of the simplified model. For $\lambda \to \infty$, the behaviors of both the exponential and rectangular memory profile models are similar to that of the traditional RW. However, for $\lambda \to 0$, both models behave like the ERW model.

In conclusion, we have shown that it is possible for a random walk to have exponential memory decay and still display anomalous diffusion. The necessary ingredient is that the exponential decay constant must be time-dependent. The reciprocal of the decay constant has units of time, and indeed it quantifies a characteristic time scale (e.g., similarly to the mean lifetime in radioactive decay). In the above models this time scale grows linearly with time, so that even at long times, there are significant correlations for small $\lambda$. Therefore the random walk is never able to make the crossover to a Markovian regime. This nuance...
for allowing order even with exponentially decaying correlations is actually a general idea and is not restricted to random walks. Consider again the example of the 1D Ising model mentioned earlier. Assuming that the correlations decay approximately as $\sim \exp[-\beta J N]$, where $\beta = 1/k_B T$, we can estimate the correlation length to be of the order of magnitude of $(\beta J)^{-1}$. We can thus find a temperature that is sufficiently low for such a finite system to be significantly correlated, i.e., to have order of the scale of the system size. However, if we allow ourselves to lower the temperature $T$ (i.e., increase $\beta$) as we increase the system size while maintaining $\beta J/N$ constant, then we can keep the spins correlated no matter how large the system is. In the thermodynamic limit, of course, the temperature necessary to sustain order over the entire system drops to $T = 0$, as it should. However, for any finite system there is a positive temperature at which the spins become significantly correlated.

We close by establishing clearly the necessary ingredient to have anomalous diffusion in random walks whose step sizes have finite variance. (Infinite variance of steps can lead to Lévy flights, which is a distinct and separate deep topic, beyond the scope of this article.) We have already shown that explicit power law decay is not a necessary ingredient.

As is well known in the literature, the necessary ingredient is for some characteristic time scales of the random walk to diverge in time. In fractional Brownian motion [25, 26] and in subdiffusive continuous time random walks, for example, the correlation distances become infinite. Similarly, a non-renewal type of process, resulting from correlations in waiting times or jump lengths in continuous time random walks (CTRWs), can lead to such effects. Such non-renewal components are seen in studies of a recent generalization [27, 28] of random walk models, leading to anomalous diffusion of both sub- and superdiffusion type. In the exponential memory model studied above, the inverse decay constant $t/\lambda$ is linear in time and hence diverges. The central limit theorem cannot therefore be applied, even for the renormalized random walk model.

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