Inferring Lévy walks from curved trajectories: A rescaling method

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An important problem in the study of anomalous diffusion and transport concerns the proper analysis of trajectory data. The analysis and inference of Lévy walk patterns from empirical or simulated trajectories of particles in two and three-dimensional spaces (2D and 3D) is much more difficult than in 1D because path curvature is nonexistent in 1D but quite common in higher dimensions. Recently, a new method for detecting Lévy walks, which considers 1D projections of 2D or 3D trajectory data, has been proposed by Humphries et al. The key new idea is to exploit the fact that the 1D projection of a high-dimensional Lévy walk is itself a Lévy walk. Here, we ask whether or not this projection method is powerful enough to cleanly distinguish 2D Lévy walk trajectories from a simple Markovian correlated random walk. We study the especially challenging case in which both 2D walks have exactly identical probability density functions (pdf) of step sizes as well as of turning angles between successive steps. Our approach extends the original projection method by introducing a rescaling of the projected data. Upon projection and coarse-graining, the renormalized pdf for the travel distances between successive turnings is seen to possess a fat tail when there is an underlying Lévy process. We exploit this effect to infer a Lévy walk process in the original high-dimensional curved trajectory. In contrast, no fat tail appears when a (Markovian) correlated random walk is analyzed in this way. We show that this procedure works extremely well in clearly identifying a Lévy walk even when there is noise from curvature. The present protocol may be useful in realistic contexts involving ongoing debates on the presence (or not) of Lévy walks related to animal movement on land (2D) and in air and oceans (3D).

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I. INTRODUCTION

Many kinds of particles (e.g., self-propelled [1–4]) are believed to diffuse anomalously and some are known to perform Lévy walks or similar movement patterns. The knowledge of the type of movement, diffusion, and transport is of fundamental importance in the study of a variety of phenomena [5–7]. In this context, analyzing and inferring a Lévy walk from trajectory data is a crucial step in the analysis of empirical data sets. Such analysis is a relatively easy task in one dimension (1D), because changes in the direction are unambiguous to identify and the asymptotic power-law distribution of traveled distances can be readily obtained through standard statistical data analysis [8,9] (e.g., histograms of traveled distances). Specifically, in 1D the points at which the velocity changes sign define unambiguously the traveled distances. If the probability density function (pdf) of these distances has a (truncated) power-law tail, then one can confidently conclude that there is a Lévy process. What makes this task so easy is that there is no curvature in 1D. However, in 2D and 3D the velocity is a vector rather than a scalar, so turnings can be continuous rather than localized. Without clearly localized turning points it is impossible to unambiguously estimate the pdf of the distances between turnings. Indeed, the very idea of “distance between turning points” becomes ill-defined and nebulous. For this reason, path curvature makes the clear identification of a Lévy walk very much harder [10–12]. In fact, there has been heated debate on how to correctly analyze and infer Lévy walks for 2D and 3D empirical data [13–21].

Recently, a new method [22] for detecting Lévy walks from empirical (or simulated) data sets has been proposed. It involves the analysis of the 1D projections of the higher dimensional curved trajectories. The key new idea was to exploit the fact that the 1D projection of a high-dimensional Lévy walk is itself a Lévy walk.

Traditionally, mathematically defined Lévy walks in 2D and 3D do not have curvature. Rather, they consist of a sequence of concatenated straight path segments. The turning points are thus unambiguously located at the junctions between the concatenated straight-line segments. Since empirical data always has noise, such idealized Lévy walks are not observed empirically. Typically, actual path trajectories include curvature to some extent, either due to measurement error or to real fluctuations in the direction of the velocity. Accordingly, it makes sense to generalize the traditional concept of Lévy walks to allow for curvature. Indeed, contrary to conventional wisdom, curvature is not the exception, but rather the norm. We will thus use the term Lévy walk in what follows in a broader sense, to include curved trajectories that in every way except for curvature behave just like traditional Lévy walks.

The projection method for identifying Lévy walks in 2D and 3D is especially relevant in the context of the heated
debate about whether or not animals perform Lévy walks while searching, e.g., for food or mates [23, 24]. Though there is compelling empirical evidence that Lévy walks correctly describe important aspects of actual animal motion of diverse species in a broad variety of environments and circumstances (see, e.g., Refs. [22–25]), there are still several arguments being used against the idea of Lévy walks [26]. One criticism deals with the statistical data analysis and statistical inference. In this context, much progress has been made [15, 22, 25, 27–30] in applying more sophisticated methods to study empirical data sets, such as maximum likelihood estimation (MLE), pdf parameter estimation in combination with goodness-of-fit (GOF) tests, and model selection through Akaike or Bayesian weights [31]. Moreover, the objective identification of the step lengths in curved paths in 2D and 3D has been hindered by the use of procedures that strongly rely on ad hoc choices of parameters [13, 16, 18–20]. This is the so-called discretization problem: of the many ways that one can discretize a continuous path, which is the best or good enough? These difficulties, in particular, make it hard to correctly distinguish between Lévy walks and correlated random walks (CRWs) in 2D and 3D [14]. Whereas in the former superdiffusivity extends over all scales, in the latter the presence of finite-range correlations can give rise to a transient superdiffusive dynamics up to scales comparable to the correlation length. So Lévy walks and CRWs can lead to very similar dynamics and visual appearance on specific spatial and temporal scales [10].

Here we apply the general ideas of the newly proposed projection method [22] to two different models. The two 2D random walks share the same distributions of step sizes and turning angles. One is a genuinely superdiffusive random walk with long-range correlations built by adding curvature on to a backbone of a traditional Lévy walk. The second model is a short-range correlated random walk (CRW). In Ref. [22], step lengths smaller than a minimum cutoff are simply removed from the data set (to minimize artificial effects resulting from the 1D projection). Here we introduce a new way to deal with the small step-length regime and the presence of correlations acting over distinct scale ranges. We apply a rescaling or coarse-graining procedure that degrades information about the small step-length regime, retaining information only on large-scales. We report that the projection method combined with this renormalization protocol works extremely well in clearly discriminating short-range CRWs from curved Lévy walks. The method is so successful that it is able to clearly distinguish between them even when the two original 2D trajectories are described by identical turning angle distributions as well as identical step length distributions. The projection method discussed here cannot give a false positive: if the 1D projection is a Lévy process, then the underlying 2D process is a Lévy process at least on the chosen axis.

II. METHOD AND MODELS

Lévy flights and walks form a class of random walks leading to anomalous diffusion [23]. Whereas in Brownian motion the density PDF $P(\ell)$ of step lengths $\ell$ has finite statistical moments. In a 1D Lévy walk, the step or “flight” lengths $\ell$ has asymptotic power-law decay, $P(\ell) \sim \ell^{-\alpha}$, resulting in divergences of statistical moments of order $\geq \mu - 1$. If the random steps are uncorrelated—or even short-range correlated [32]—the central limit theorem (CLT) applies only when the exponent $\mu$ is large enough for the variance of the pdf to be finite. In 1D space, Brownian motion arises due to the CLT for $\mu > 3$, whereas the variance becomes infinite in the range $1 < \mu < 3$. In the latter case the generalization of the CLT leads to the family of $\alpha$-stable Lévy distributions, with the Lévy parameter $\alpha$ related to the power-law exponent through $\alpha = \mu - 1$ (values $\alpha < 0$ or $\mu \leq 1$ do not correspond to normalizable pdfs). These features also generalize to higher dimensions, although in this case the analysis of the exponent $\mu$ must be handled with some care [23].

From the above, we see that an appropriate strategy to infer a Lévy walk from empirical or simulation data in 1D consists of inspecting its pdf of step sizes. Data analysis performed using inference methods [22, 28, 30] certainly would be able to identify a “pure” (e.g., generated without distortions or deviations) Lévy random walk behavior. For empirical data, however, the analysis can be quite problematic even in 1D due to limitations [22] related to diverse technical complications and/or the presence of some intrinsic noise in the data set (e.g., some short-range correlated Brownian motion might be superimposed on the Lévy walk pattern). In fact, in some instances the presence of the noise can even “chop up” the ultralong steps, which are essential to characterize the Lévy walk. This “chopping up” effect can arise even without noise. Consider the following illustrative example of a Lévy walk of constant speed. Data showing an ultralong step or flight of 1000 distance units can be interpreted equivalently as 1000 consecutive steps of unit length taken all in the same direction. If the discretization procedure used on the empirical data only considers steps of unit length, then a Lévy walk pdf will no longer have the power-law tail. Instead, long-range power-law temporal correlations will appear in this representation of the data. The long steps become “chopped up.” On the other hand, if the data is discretized according to the rule that a “step” is the distance between turnings, then the step length pdf of the Lévy walk will clearly have a power-law tail. But in this representation, there will be temporal anticorrelations, since at every turning the velocity changes sign. So even in 1D detecting or inferring a Lévy walk depends on how the trajectory is represented.

In 1D some of these difficulties can be circumvented by applying a rescaling of the data set, then reconstructing the pdf of step lengths. Since the noise and other artifacts have specific scales, they will attenuate under coarse graining. Hence, we can focus on the large scale regime and not on the fine-scale details of the stochastic process. In other words, finite-range correlations are swept out upon a suitable rescaling, so that a 1D CRW with short-term superdiffusive dynamics will appear as a simple Brownian walk after this procedure [33, 34]. In contrast, a true 1D Lévy walk pattern has scale-free properties that subsist under coarse graining. This means that the self-affine properties of Lévy walks guarantee that it will appear as such even after rescaling [23, 24]. Lévy walks look like Lévy walks even after coarse-graining, whereas CRWs look like Brownian motion upon coarse-graining.

Unfortunately this idea cannot work in 2D or 3D without modifications. The presence of curvature in higher-dimensional random walks poses challenges that are not
present in the 1D counterpart and that are associated with the discretization problem \cite{13,16–20}. Indeed, how can a curved path be properly segmented into discrete random walk steps? There is no objective criterion to identify where one step ends and the subsequent begins. In this context, a recently proposed method \cite{22} has made significant progress. It essentially exploits the fact that the projection onto 1D of a 2D or 3D Brownian (Lévy) walk is itself a Brownian (Lévy) walk. The 1D projection cannot contain any curvature. So an \(n\)-dimensional trajectory can be studied as \(n\) separate 1D trajectories. Then the usual methods for studying 1D Lévy walks can thus be successfully applied to the projected trajectories.

In Ref. \cite{22} this key idea is used to detect Lévy walks in both empirical and numerically generated data sets. However, the results in Ref. \cite{22} show that even for numerically controlled walks, the 1D projection introduces a number of step sizes artificially smaller than the minimum actual length \(\ell_{\text{min}}\). From a practical point of view, this is a drawback. Whereas in simulated walks (e.g., used as a comparison standard) the value of \(\ell_{\text{min}}\) is previously fixed, in empirical data sets it cannot be known \textit{a priori}. Then, this must be overcome by applying an objective method for the estimation of \(\ell_{\text{min}}\) \cite{30}. In any case, the pdfs of the 1D projected steps generally display two domains: a short-range regime prevailing these “unwanted” steps and described by the Brownian statistics, and the complementary domain that properly reflects the statistical properties of the original, i.e., higher-dimensional, random walk (according to the above discussion). Therefore, to avoid the spurious influence of these artificially small step sizes, in their method these are simply removed from the data set \cite{22} before performing the utmost 1D statistical analysis.

In the present work we test the projection method using synthetic (i.e., artificial) data. We also introduce a new way to deal with the short-range domain of the 1D projections of 2D and 3D random walks. That is, instead of just eliminating from the data set all 1D projected steps of length smaller than \(\ell_{\text{min}}\), we apply the rescaling procedure discussed above, which preserves the long-range statistical properties of the actual random walks in 2D and 3D. Essentially, we resample the trajectory at a lower resolution. In the language of statistical mechanics, we coarse-grain the projected trajectory. This resampling eliminates all information about the trajectory at scales below the resampling scale (this is a well-known effect associated with the Nyquist sampling theorem \cite{35}).

To check the ability of the 1D projection with rescaling method to identify Lévy walk behavior of higher-dimensional random walks, we test it against two distinct 2D curved random walks \cite{14,36}, defined below.

\subsection*{A. Model I}

The first model is a curved 2D Lévy walk. Initially, a 1D traditional Lévy walk of total length \(L = 10^5\) is generated, with step sizes \(\ell_j \geq \ell_0 = 1\), for \(j = 1, 2, \ldots, N\). In this process, steps larger than 10% of \(L = 10^5\) (i.e., \(\ell_j > 10^5\)) are summarily truncated, so that the statistics is not dominated by a single step. The resulting truncated Lévy walk is known \cite{37} to retain the general statistical properties of actual (i.e., nontruncated) Lévy walks to a considerable extent \cite{38}.

This sequence of \(N\) step sizes is then used to generate \(N\) CRWs as follows. The \(j\)th CRW has fixed step size \(\ell_0\) and total length \(\ell_j\), so that the number of steps \(N_j\) in the \(j\)th CRW is given by the largest integer smaller than or equal to \(\ell_j/\ell_0\). The \(N_j\) turning angles are drawn from the wrapped Cauchy distribution (WCD) with width parameter given by

\[
\rho_j = (e^{-1} - \exp[-\ell_0/\ell_j]/(e^{-1}) - 1) \quad [14].
\]

This implies that the correlation length \(\xi_j\) of the \(j\)th CRW is comparable to the corresponding value of the step size \(\ell_j\) of the underlying Lévy walk. The \(N\) CRWs thus mimic \(N\) traditional Lévy walk segments. Finally, the \(N\) CRWs are joined together and the seams made smooth using the method described in Ref. [14], the technicalities of which are irrelevant for our purposes here. Thus, the total length of the concatenated walk coincides with the length of the original traditional 1D Lévy walk.

The stitched-together 2D random walk is locally a non-Markovian CRW, but globally a curved Lévy walk. Originally, the definition of Lévy walks did not contemplate curvature \cite{39}. Here, curvature in Model I arises from the WCDs of turning angles. The close connection between the
sets of correlation lengths \( \xi_j \) of the WCDs and the step sizes \( \ell_j \) of the underlying Lévy walk model implies that Model I indeed corresponds to a genuinely superdiffusive curved random walk with long-range power-law correlations and scale-free properties [14,36,40].

### B. Model II

Model II is a CRW. Model II is generated by simply shuffling the turning angles of model I, thereby destroying completely all correlations. Consequently, models I and II have exactly the same distribution of turning angles and step sizes, although long-range correlations are present only in model I. Specifically, the theoretical step-size distribution for both models is a Dirac \( \delta \) function, with all steps having unit length \( \ell_0 = 1 \). The turning angle distribution for both models is the WCD. The only difference between models I and II is that the turning angles are (long-range) correlated in model I but uncorrelated in model II.

### C. Difficulty of distinguishing the two models

The crucial point is precisely the impossibility to distinguish models I and II just from the pdfs of step sizes and turning angles. The absence of correlations in the Markovian model II makes it essentially diffusive (Brownian) rather than superdiffusive, i.e., without the scale-free properties typical of Lévy walks. Consequently, the 2D curved paths generated by models I and II have similar visual appearance on very short scales (or for a quite small number of steps), but differ noticeably at larger scales (see Fig. 1). Although the directional memory in model II has short range, due to the shuffled turning angle distribution, the long-range character of model I follows closely the underlying Lévy walk backbone.

We choose for the underlying Lévy walk of model I the superdiffusive power-law exponent \( \mu = 2 \) and the minimum step length \( \ell_0 = 1 \). For both models we have generated 2D random walks with total length \( L = 10^5 \). These models are shown at two different scales in Fig. 1. Note that at small scales, the two walks are similar. On the other hand, their large scale behavior is quite different.

### III. RESULTS

We now apply the projection method [22] combined with the rescaling ideas discussed above to 2D curved paths generated using models I and II. As described in the preceding section, these random walk models possess identical pdfs of step sizes and turning angles, which makes their differentiation more difficult and basically relying on the analysis of the directional correlations alone.

Each generated walk is associated with a data set in the form \( \{t_n, x_n, y_n\} \), where \( n = 1, 2, \ldots \) denotes the temporal ordering of the steps, and \( x_n \) and \( y_n \) are their coordinates in the \( xy \) plane. Since the step lengths in models I and II are originally set to \( \ell_0 \), then \((x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2 = \ell_0^2\).

We define the parameter \( \lambda \) for the rescaling procedure. Rescaling is introduced when the time series is resampled considering points separated by a time interval \( \lambda \) in the original data sets. The value \( \lambda = 1 \) correspond to the original data sets (without rescaling).

Figure 2 shows the joint pdf in \( x \) and \( y \) directions of step length vectors (or velocity vectors) for \( \lambda = 1 \) Fig. 2(a) (original data sets), and rescaled data using (b) \( \lambda = 20 \) and (c) \( \lambda = 100 \). In the original data set (a) all step lengths are fixed at \( \ell_0 = \ell_0 = 1 \) for both models I (left column) and II (right column). As rescalings under larger values of \( \lambda \) are applied, the difference between the distributions of models I and II becomes remarkable. The presence of long steps in model I is indicated by the dense filling close to the circle edge. In contrast, the rescaled data of model II is similar to a Gaussian, with a high density of short steps and a rapidly decreasing probability for large displacements. The important point to note is that the rescaling (or renormalization) brings out the difference between the two models. Without the rescaling, the models look indistinguishable, but after rescaling the models can be easily distinguished.
possible rescaled step length, $\lambda \ell_0$. The filling of this region gives a visual indication of the step-lengths distribution under rescaling. Indeed, note that all step lengths are fixed at $\lambda \ell$ and II. When an intermediate rescaling of $\lambda = 20$ is applied, the difference between the distributions already becomes remarkable, with the data of model I still more likely to lie close to the circle edge, whereas model II presents points distributed all over the circular region. Even for a larger rescaling using $\lambda = 100$, the distribution for model I still contains many long steps located close to the circle boundaries, in contrast with the Gaussian-like distribution of model II, with a high density of small steps and a rapidly decreasing probability for large displacements.

Figure 2 indicates that, in principle, a rescaling using $\lambda = 20$ could be sufficient to differentiate between models I and II. We next quantify this difference.

Figure 3 displays the rms length, $\ell_{rms} = \langle (\ell^2 - \langle \ell^2 \rangle)^{1/2} \rangle$, of models I and II as a function of the rescaling parameter $\lambda$. In model II the crossover to the Brownian behavior, $\ell_{rms} \sim \lambda^H$, with $H = 0.5$ (red line as a guide to the eyes), starts around $\lambda \approx 50$. Below this value a transient superdiffusive behavior is set. In contrast, the superdiffusivity of model I, with $H > 0.5$, does not show any signal of convergence to the Brownian regime in the interval studied. The rescaling method thus yields the values of the exponent for the two models. The models are easily distinguished by their exponents. Only model I is superdiffusive.

In order to be able to obtain good enough statistics on step lengths or more generally distances, we will use the projection method to make it easier to identify turning points unambiguously, as explained previously. Then we will be able to tell a Lévy walk apart from other kinds of superdiffusive walks.

The projection onto the $x$ and $y$ axes are two 1D walks, denoted by the two time series $(t_n, x_n)$ and $(t_n, y_n)$ [see Fig. 4(a)]. From these 1D projection data sets, two pdfs of 1D flight lengths are generated for each model, one for $x$ axis and the other for $y$ axis, plotted in the histograms of Fig. 5.

We note that, when the original scale structure is considered, there is already some difference between the histograms for models I and II, in Figs. 5(a) and 5(b). This difference is due to how the correlation properties are different in models I and II. Model I has long-range correlations, whereas model II has only short-range correlations since it is a Markov process.

At this stage, the original projection method [22] requires the analysis of the minimum length $\ell_{min}$, below which all points in the data sets $(t_n, x_n)$ and $(t_n, y_n)$ should be removed. However, in contrast with such a prescription, instead we perform a rescaling of the data sets, aiming to infer the presence of the Lévy pattern in model I.

In Fig. 4(b) we reconstruct the time series by considering points separated by an interval $\lambda = 100$ in the original data sets. Figure 4(b) displays the rescaled sets in which, e.g., the former point $t_n = 200\lambda = 20\,000$ in Fig. 4(a) now corresponds to the point $t_n = 200$, which in turn is connected with the point $t_n = 201$, formerly $t_n = 201\lambda = 20\,100$, and so on. The insets of Fig. 4 show this correspondence in detail.

Under the coarse graining (introduced by the scale $\lambda$) the pdfs of the rescaled projected flight lengths are now computed. Figure 6 shows that the difference between the rescaled pdfs associated with models I and II is very clear. The tail enhances considerably under the rescaling procedure for the model I.

We also observe that the rescaled projected lengths of model II become restricted to a much narrower range than those of model I. Consistently, a Gaussian fit works well only to the pdfs of model II (see the dashed lines in Fig. 6 and in the semilog plots of Fig. 7). This indicates that the rescaling under $\lambda = 100$ has swept out any possible signal of short-term correlations in model II. In contrast, the long-range power-law correlations of model I are kept essentially intact under this procedure, as a consequence of its backbone Lévy structure.

The outcome of this rescaling process is similarly appreciated in the double-log plots shown in Fig. 8. Whereas the original (i.e., not rescaled, $\lambda = 1$) pdfs of projected flight lengths display similar behavior for both models I and II, they are distinct under coarse graining (i.e., for $\lambda = 100$), with the rescaled data from model I following approximately a power-law decay, in contrast with the much more limited range of the projected flight lengths of model II. The signature of Lévy walks, which are the power-law tails in the pdf, can be seen in Fig. 8.

In addition to the visual indication of power-law scaling seen in Fig. 8 for model I but not for model II, one can also try to obtain estimates of the power-law exponents from the 1D projections for model I. For $\lambda = 1$ (without rescaling), the tail of the histogram of the absolute value of the flight lengths of Brownian motion, for example, can be superdiffusive but is not a Lévy walk or flight [23].
FIG. 4. (Color online) Evolution with the number of steps $t_n$ of the coordinates $(x, y)$ of the 2D curved random walks shown in Fig. 1(b) generated using models I and II: (a) original (i.e., nonrescaled, $\lambda = 1$) data sets; (b) rescaled data using $\lambda = 100$. The difference between (a) and (b) lies mainly in the timescale. The insets show details of the correspondence under rescaling between parts of the plots (a) and (b).

FIG. 5. (Color online) Histograms of distances between successive turnings projected onto the axes $x$ [(a) and (c)] and $y$ [(b) and (d)] of the time series shown in Fig. 4(a) (nonrescaled data, $\lambda = 1$). (a), (b) Model I; (c), (d) model II. Practically no difference is observed. This lack of difference demonstrates the difficulty of distinguishing the two models.
FIG. 6. (Color online) Histograms of the distances between successive turnings projected onto the axes x [(a) and (c)] and y [(b) and (d)] of the time series shown in Fig. 4(b), now rescaled data using $\lambda = 100$. (a), (b) Model I; (c), (d) model II. The Gaussian fits, shown in dashed lines, describe model II reasonably well, but completely fail at the tails for model I. The rescaling has led to a noticeable difference between the models, when compared with the nonrescaled analysis shown in the previous figure.

FIG. 7. (Color online) Semilog plots of the histograms displayed in Fig. 6 for the rescaled data using $\lambda = 100$, including the best fits to a Gaussian pdf shown in dashed lines. (a), (b) Model I; (c), (d) model II. Notice how model I has heavy tails, whereas model II is basically Gaussian.
FIG. 8. (Color online) Double-log plots of the histograms displayed in Figs. 5 and 6 for (a), (b) the original (i.e., nonrescaled, $\lambda = 1$) data, and (c), (d) the rescaled data using $\lambda = 100$. Black circles (blue squares) depict the results for the model I (model II). Notice how models I and II are very different at the tails, which are approximately described by a (truncated) power law. This difference only arises due to the rescaling. Without rescaling, the models are indistinguishable, as shown in Fig. 5. The key point to note is that the projection method together with the rescaling method makes it possible to reveal that model I is a (curved) Lévy walk.

the 1D projections has exponent $\mu \approx 1.7$ (obtained using nonlinear fitting for flight lengths ranging from 10 to 200 units). For $\lambda = 100$ we obtain $\mu \approx 2.1$ (for flights ranging from 100 to 1000 units). Figure 3 suggests an independent method for estimating the exponent. If we assume that the quantity $l_{\text{rms}}$ scales with $\lambda$ with exponent equal to the widely used Hurst exponent $H$, we can use the scaling relations $H = 1/(\mu - 1)$ for Lévy flights or $H = (4 - \mu)/2$ for Lévy walks [23]. From Fig. 3 we obtain the estimate $H \approx 1.1$, from which we obtain $\mu \approx 1.9$ (assuming Lévy flights) and $\mu \approx 1.8$ (assuming Lévy walks). In contrast to Lévy flights, however, Lévy walks are restricted to $H \leq 1$ (see Ref. [23] for details).

If we thus assume the maximum allowed value $H = 1$ we obtain $\mu = 2$ exactly for a Lévy walk. Notwithstanding the encouraging agreement with the correct value $\mu = 2$, we note that the estimation of power-law exponents from trajectory data is not straightforward and the topic lies just beyond the scope of the present discussion.

IV. CONCLUSIONS

As shown above, the clear identification of a Lévy walk pattern in 2D and 3D curved paths is recognizably much more difficult than in 1D. One of the main difficulties is related to the technical aspect of segmenting the curved trajectory into discrete steps. This issue was successfully addressed through the recently proposed projection method [22]. Additionally, the presence of finite-range correlations in the curved walk might considerably hinder the clear distinction between a CRW and a Lévy walk. The difficulty increases further if the walks have completely identical pdfs of step sizes and turning angles, making their differentiation possible only through the analysis of the directional correlations.

In this work we have addressed precisely this case by studying two 2D curved random walk models, I and II, defined in Sec. II. We have introduced a modification to the original projection method in which, instead of simply discarding projected step lengths smaller than a minimum cutoff [22], we perform a rescaling of the projected data sets and reconstruct the step lengths pdfs. This allows us to test the statistical properties of the correlations under coarse graining. Finding the useful range of $\lambda$ plays more or less the same role as finding $l_{\text{min}}$ in Ref. [22]. We show that these combined procedures work extremely well to distinguish between the short-range correlated walk behavior of model II and the Lévy pattern of model I resulting from the long-range directional memory.

Summarizing, the approach here has shown to be a very valuable tool in characterizing ubiquitous curved random walks in 2D and 3D spaces. So, it certainly can help to settle common disputes about the presence or not of Lévy walks in actual empirical data, such as in animal foraging.

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