Landscape-scaled strategies can outperform Lévy random searches

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Information on the relevant global scales of the search space, even if partial, should conceivably enhance the performance of random searches. Here we show numerically and analytically that the paradigmatic uninformed optimal Lévy searches can be outperformed by informed multiple-scale random searches in one (1D) and two (2D) dimensions, even when the knowledge about the relevant landscape scales is incomplete. We show in the low-density nondestructive regime that the optimal efficiency of biexponential searches that incorporate all key scales of the 1D landscape of size $L$ decays asymptotically as $\eta_{\text{opt}} \sim 1/\sqrt{L}$, overcoming the result $\eta_{\text{opt}} \sim 1/(\sqrt{L}\ln L)$ of optimal Lévy searches. We further characterize the level of limited information the searcher can have on these scales. We obtain the phase diagram of bi- and triexponential searches in 1D and 2D. Remarkably, even for a certain degree of lack of information, partially informed searches can still outperform optimal Lévy searches. We discuss our results in connection with the foraging problem.

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I. INTRODUCTION

Random searches occur whenever full information on the location of target sites is not available to the searcher [1–6]. This situation is so common that realizations of the random search problem are widely found in contexts as diverse as animal foraging, information technology, and proteins searching for DNA sites [1,2].

The degree of knowledge about the search space is key to the search performance. On the one side, in the complete absence of information it is long known [7] that the maximum search efficiency is achieved by a random search walker with step lengths distributed according to the Lévy $\alpha$-stable distribution. In the so-called nondestructive or asymmetric regime, the walker restarts the search in the vicinity of the last target found and can revisit the targets many times or else it can find new targets closeby and deplete a patch (e.g., in heterogeneous landscapes). In this case, the optimal search strategy corresponds to the Lévy index $\alpha \to 1$ (Cauchy limit) in the scarce regime of low target density [7–13].

On the other side, if the agent has full information on the targets locations, then the so-called traveling salesman problem yields an upper limit to the efficiency of the deterministic optimal path solutions [14].

Many realistic contexts to which the random search problem applies fall between these extremes, with the searcher possessing only partial information on the landscape. In this case, it is not trivial to identify what type of information helps to unequivocally enhance the search efficiency if compared to the fully uninformed scenario. Local temporal or spatial information are not always a guarantee of improved search performance [15–17]. However, the situation may be quite distinct when the partial information has global character, as in the knowledge of the key landscape scales (e.g., mean free path to targets).

Here we show numerically and analytically that, even when the knowledge on the key landscape scales is incomplete, partially informed multiple-scale random searches can outperform the paradigmatic uninformed optimal Lévy searches. We study the hyperexponential distributions of step lengths in one (1D) and two (2D) dimensions, which in recent years have been successfully applied to model animal foraging [18–21]. For nondestructive searches in a 1D landscape of size $L$, when the typical scales of the biexponential distribution are tuned to the proper global scales, we derive that the asymptotic optimal efficiency, $\eta_{\text{opt}} \sim 1/\sqrt{L}$, exceeds the result $\eta_{\text{opt}} \sim 1/(\sqrt{L}\ln L)$ of optimal Lévy searches with $\alpha \to 1$. However, the situation is much less clear in the partially informed scenario. In this case, we obtain the phase diagram of bi- and triexponential searches in 1D and 2D. Remarkably, in a large range of the parameters space these multiscale searches can still outperform the optimal Lévy searches. Our results advance the understanding of improved searching based on...
(partially) informed strategies, as in the relevant ecological problem of animal foraging at the landscape level [22].

II. INFORMED AND UNINFORMED HYPEREXPONENTIAL SEARCHES

Consider initially a random walker that starts at time \( t = 0 \) from the position \( x_0 \) in a 1D finite domain \( 0 \leq x \leq L \). Target sites are set at the boundaries \( x = 0 \) and \( x = L \). The probability density function to perform a step of length \( \ell \) is \( p(\ell) \). We assume constant velocity \( v \), so that \(|\ell| = \nu t\) where \( t \) is the step duration. We also consider moves to the right and to the left to be equiprobable, \( p(|\ell|) = p(-|\ell|) \). Denoting \( f(t, x_0) \) as the first passage time (FPT) distribution to pass through any boundary for the first time, the mean FPT (MFPT) is \((T)(x_0) = \int_0^\infty t f(t, x_0) dt\), which by considering the Laplace transform \( F(s, x_0) \) can also be written as \((T)(x_0) = -\lim_{s \to 0} D F(s, x_0)/ds\). The search efficiency, defined [7] as the number of targets found divided by the total path length, is \( \eta(x_0) = 1/(\langle D \rangle) = 1/(\nu(T)) \), where \((\langle D \rangle)(x_0)\) is the mean distance traversed to the first encounter of a target.

Before considering the hyperexponential distributions of step lengths, we review the method of Laplace and Fourier transforms [23–25] by first analyzing searches with single exponential \( p(\ell) = \exp(-|\ell|/d)/(2d) \). Denoting by \( P(x, t; x_0) \) the probability density to be at \( x \) in time \( t \) with periodic boundary conditions (PBC),

\[
P(0, t; x_0) = f(t, x_0) + \int_0^t f(t - t', x_0) P(0, t'; 0) dt'.
\]

\( P(x, t; x_0) \) in the domain \( 0 \leq x \leq L \) with PBC can be found from \( P_0(x, t; x_0) \) in free space through the method of images [23], \( P(x, t; x_0) = \sum_{n=-\infty}^{\infty} P_0(x + nL, t; x_0) \). From the Laplace and Fourier transforms of (1) the MFPT obtained is in agreement with Ref. [26], as expected, so that the efficiency of single exponential searches is

\[
\eta(x_0) = \frac{2}{L + x_0(L - x_0)/d}.
\]

The above \( \eta \) does not display a maximum for fixed \( x_0 \) and \( L \) and finite \( d \). In the nondestructive or asymmetric regime, with the searcher starting from the close vicinity of a target, \( x_0/L \to 0 \), the asymptotic decay with \( L \) for single exponential searches reads

\[
\eta \sim \frac{1}{L} \quad \text{(single exponential)}.
\]

A much more interesting scenario emerges when extra (possibly competing) scales are added to the search problem. So we turn to the hyperexponential searches,

\[
p(\ell) = \sum_{i=1}^{N} \omega_i \exp(-|\ell|/d_i)/(2d_i),
\]

with \( N \) length scales \( d_i \) and the corresponding statistical weights \( \omega_i \), where \( \sum_{i=1}^{N} \omega_i = 1 \). We term an \textit{uninformed} hyperexponential search walk when its parameters set \([d_i, \omega_i]\) displays no significant relation with the key length scales of the random search problem, e.g., \( x_0 \) and \( L \) in 1D. On the other hand, if somehow the searcher can tune at least a fraction of \([d_i, \omega_i]\) in terms of the key scales to improve the search performance, then we call it an \textit{informed} hyperexponential walk.

The limit \( N \to \infty \) with a proper set \([d_i, \omega_i]\) in Eq. (4) turns the hyperexponential walk into a Wiener process with a hierarchy of self-similar clusters [27] recently applied to study animal foraging [20].

The above procedure to calculate \( \eta \) of the single exponential \( p(\ell) \) can be extended [25,28] to treat the hyperexponential case. One has to take into account that the first passage through a boundary can take place along a step drawn from any \( x \)th exponential in Eq. (4). The associated FTP distributions \( f_i(t, x_0) \) thus imply the MFPT \((T)(x_0) = -\lim_{s \to 0} \sum_{i=1}^{N} dF_i(s, x_0)/ds\). For instance, the corresponding probability that the walker, passing through position \( x \) at time \( t \), is performing at that instant a flight drawn from the exponential \( i \) in Eq. (4) is given by an equation similar to (1) but defined for \( P_i(0, t; x_0) \) [25].

Hyperexponential search walks with efficiency in the nondestructive regime comparable with that of the Lévy optimal search should allow the access to both boundaries with relatively high probability. This can be achieved with at least one characteristic length \( d_i \gtrsim L \) in Eq. (4). Take, for instance, the biexponential search walk with \( N = 2 \) [25]. In the nondestructive regime, \( x_0/L \to 0 \), the encounter of the faraway (close) target at an initial distance \( L - x_0 \) is favored by considering, say, \( d_1 \gg L (d_2 \ll L) \), so we obtain

\[
\eta(x_0) = \frac{2d_2 \sqrt{\omega_1}}{L \omega_1 [d_2 + x_0 (1 - \sqrt{\omega_1})] + 2d_2 (1 - \sqrt{\omega_1})(d_2 + x_0)}.
\]

Differently from Eq. (2), the efficiency \( \eta \) in (5) displays a maximum for \( x_0 \) and \( L \) fixed and finite \( d_2 \) and \( \omega_1 \). In the nondestructive regime its asymptotic behavior in the \textit{uninformed} biexponential search walks, \( \eta \sim 1/(\sqrt{\omega_1} L) \), is essentially the same as that of the single exponential walks, Eq. (3). However, a drastic change takes place when it is allowed for the searcher to tune its parameters set to improve the search efficiency. Indeed, by maximizing \( \eta \) in Eq. (5) in the nondestructive regime, we find that the optimal choices \( d_2 \sim x_0 \) and \( \omega_1 \sim 2x_0/L \) for \textit{informed} biexponential search walks imply

\[
\eta_{\text{opt}} \sim \frac{1}{\sqrt{L}} \quad \text{(biexponential)}.
\]

This decay is slower than both Eq. (3) and the uninformed biexponential case.

The mechanism underlying this result arises from the compromise balance in \( p(\ell) \) of an exponential \((i = 1)\) with large steps that can occasionally take the searcher to distances as far as \( L \) (high \( d_1 \) and low \( \omega_1 \)) and another exponential \((i = 2)\) that favors intensive searches at nearby distances \((d_2 \ll L)\). A similar balance is also possible with informed hyperexponential searches with \( N \geq 3 \) in 2D, see below.
III. LÉVY SEARCHES

Lévy walkers with left-right symmetry, \( p(|\ell|) = p(-|\ell|) \), present nonskewed distribution [29]

\[
p(\ell) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-|k|^\alpha - ik\ell},
\]

with the stability index \( \alpha \in (0, 2] \) and \( c > 0 \) a scale parameter. In the nondestructive case, a classic result \([7,8]\) states that uninform Lévy searches with maximum efficiency display optimal solution \( \alpha \to 1 \) for \( L \to \infty \).

A number of technical difficulties arise when applying the above method to Lévy search walks. First, the fact that \( p(\ell) \) generally lacks closed-form expressions based on elementary functions [29] hinders significantly the Laplace transforms that yield the FPT distribution. This drawback is not overcome even with the power-law asymptotic limit of (7), \( p(\ell) \sim c^\alpha/|\ell|^\alpha+1 \), for \( \alpha \in (0, 2) \) and \( c \) as the minimum step length without loss of generality to ensure the normalization of \( p(\ell) \). Moreover, the diverging second moment of \( p(\ell) \) for \( \alpha \in (0, 2) \) leads to the breakdown of the method of images [30].

Nonetheless, the problem of calculating the MFPT of a Lévy search walker is equivalent [31] to that of obtaining the mean distance \( \langle D(x_0) \rangle \) traversed by a Lévy flier to the first encounter of a target. So, instead of following the evolution with \( t \) of the probability distribution of the Lévy walker, we consider the density \( \rho_\omega(x) \) of the Lévy flier to be at \( x \) after \( n \) discrete steps. We thus write \( \langle D(x_0) \rangle = \sum_{n=1}^{\infty} \rho_\omega(x_0)(D_n)(x_0) \), with the probability of finding any of the boundary targets for the first time after \( n \) steps given by \( P_n = \int_0^1 (|\rho_\omega(x) - \rho_\omega(x_0)| dx \) and the mean distance traversed by a Lévy flier that encounters a target after exactly \( n \) jumps expressed as \( \langle D_n \rangle = \sum_{m=1}^{\infty} (|l_m|) \), where \( (|l_m|) \) represents the mean single step length at the \( m \)th jump. Now, defining the integral operator \( L \) with kernel \( \rho(x-x') \), \( \langle Lg(x') \rangle(x) = \int_0^1 \rho(x-x')g(x')dx' \), so that \( \rho_\omega(x) = \langle L^n \rho_\omega(x) \rangle(x) \) and \( \rho_\omega(x) = \delta(x-x_0) \), and formally summing over \( n \), we find [32,33]

\[
\langle D(x_0) \rangle = [(I - L)^{-1}(|l|)](x_0),
\]

where \( I \) is the identity operator, \((I - L)^{-1}\) is the inverse operator of \((I - L)\), and \((|l|)(x_0)\) is the mean step length starting from \( x_0 \) in the domain \( 0 \leq x \leq L \).

From Eq. (8) in the continuous limit \( c/L \to 0 \) the leading contribution [32,33] to the efficiency of \( x_0/L \to 0 \) nondestructive Lévy and power-law searches in the range \( \alpha \in (0, 1] \) where the maximum is located reads

\[
\eta(x_0) = \frac{1 - \alpha}{2L} \left( \frac{L}{x_0} \right)^{\alpha/2} \left[ \frac{1}{\alpha B(\alpha/2, \alpha/2)} \right] \left[ \frac{1}{B(\alpha/2, 1 - \alpha/2)} \left( \frac{L}{c} \right)^{\alpha-1} \right]^{-1},
\]

where \( B(x, y) \) is the beta function. By maximizing \( \eta \) with respect to \( \alpha \) we find the optimal index \( \alpha = 1 - 6[\ln(4/e) + \ln(x_0/c)]/[\ln(L/c)]^3 \). The small shift \( \sim \ln(L/c)^{-2} \) from the asymptotic Cauchy value \( \alpha = 1 \) agrees with both the mean-field-like approach to the Lévy random search problem [7] and numerical simulations [33]. Finally, by substituting the optimal \( \alpha \) into (9), we obtain the asymptotic optimal efficiency of nondestructive Lévy searches,

\[
\eta_{\text{opt}} \sim \frac{1}{\ln L} \quad \text{(Lévy)}.
\]

The Lévy optimal efficiency (10) decays asymptotically with \( L \) faster than the one of the informed biexponential search, Eq. (6). So in the \( L \to \infty \) scarce regime of optimal nondestructive searches we conclude that

\[
\eta_{\text{single exp}} < \eta_{\text{opt Lévy}} < \eta_{\text{informed opt bi-exp}},
\]

implying that informed optimal biexponential search walks (and general informed hyperexponential walks as well, see below) can outperform optimal Lévy searches.

Figure 1 shows a phase diagram in the nondestructive regime indicating the region of the parameters space of the 1D biexponential walk where it displays efficiency higher (green, inner region) or lower (blue, outer region) than the optimal nondestructive Lévy walk. The boundary line (black) was calculated from Eqs. (5) and (9) for \( x_0 = 2, c = 0.2, L = 10^1 \), and the optimal \( \alpha \). A similar picture arises from Monte Carlo results of triexponential and Lévy nondestructive search walks in 2D.

IV. COMPARISON WITH 1D AND 2D NUMERICAL RESULTS

Numerical results in 1D can be obtained by considering (8) in the discrete space, so that the integral operator \( L \) becomes a Toeplitz matrix of transition probabilities [32–34]. The running time and memory allocation are \( O(n^2) \), so we consider...
function of the 1D system size $L$ (circles) exponential and Lévy-like power-law (squares) $p(\ell)$ as a function of the 1D system size $L$, with $x_0 = 2$ and $c = 0.2$. A good agreement is noted between numerical (filled symbols) and analytical (empty symbols) results. In the nondestructive regime the predicted asymptotic behaviors (dashed lines) show that informed biexponential walks outperform optimal Lévy walks.

Overall, our 2D results for the bi- and triexponential and Lévy-like power-law distributions $p(\ell)$ exceed that of the paradigmatic uninformed Lévy searches. Remarkably, this result holds true even for a certain degree of partial information about such scales. Our results may also concur with the suggestion discussed in Ref. [22] of behavioral selective pressures to (roughly) adjust multiple-scale searches with improved performance at the landscape level. We hope our work represents a step forward to stimulate empirical and theoretical studies of partially informed multiple-scale searches with improved performance in the broad context of random searches and related problems.

FIG. 2. Optimal efficiency $\eta_{\text{opt}}$ for the single (triangles) and biaxial (circles) exponential and Lévy-like power-law (squares) $p(\ell)$ as a function of the 1D system size $L$, with $x_0 = 2$ and $c = 0.2$. A good agreement is noted between numerical (filled symbols) and analytical (empty symbols) results. In the nondestructive regime the predicted asymptotic behaviors (dashed lines) show that informed biexponential walks outperform optimal Lévy walks.

FIG. 3. Numerical efficiency $\eta$ in 2D from Monte Carlo simulations with averages over $5 \times 10^4$ walk runs. We used $L^2 = 200^2$ with PBC, $x_0 = \sqrt{2}$, and four randomly placed targets of unit radius. Bi- and triexponentials: $d_1 = x_0$, $d_1 = 10^4 L$, with $\omega_1 = 2x_0/L$ and $\omega_2 = \omega_3$ in the latter. Lévy: $c = 0.5$.

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