Russian Mathematical Surveys

THE PLANE ISING MODEL

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In this paper the statistical sum of a plane Ising model in the absence of an external magnetic field is computed.

0. Introduction. General methods of investigation in the field of statistical mechanics are as yet insufficiently developed. Therefore, mathematical models that permit exact computations are very important. The plane Ising model [1] is the most interesting of this kind. Onsager [2] first calculated the statistical sum for this model in the absence of an external magnetic field. Since then a number of papers have been published which repeat his results by different methods and investigate this model from different points of view.[1] [3] – [14]. In this paper we also calculate the statistical sum for such a model. It differs from previous works on the subject by its greater simplicity.

In §1 a soluble problem is stated. In §2 the necessary mathematical apparatus is introduced – the integral over anti-commuting variables. The rest of the paper deals with the solution of the problem.

1. Statement of the problem. Let R_N be a part of an integral plane lattice situated within and on the sides of a square with sides of length $N-1$, parallel to the axes of coordinates and with vertices at lattice points. (R_N contains N^2 lattice points.) Further, let E_N be a real linear space of dimension 2^{N^2} and $\sigma(x)$, $x=(m, n) \in R_N$ a system of operators in E_N with the properties:[2]

$$\sigma^2(x) = I, \qquad \mathrm{sp}\,\sigma(x) = 0, \qquad \sigma(x)\,\sigma(y) = \sigma(y)\,\sigma(x) \tag{1}$$

(I denotes the unit operator in E_N). The function

$$\Xi_N(\beta) = \mathrm{sp}\, e^{\beta \sum \sigma(m,n)\,(v_1\sigma(m+1,\,n)+v_2\sigma(m,\,n+1))} \tag{2}$$

is called the statistical sum of a plane Ising model in the absence of an external field. The sum in the exponent extends over all points $x=(m, n)$ of the lattice R_N, except for points of the type (N, n) and (n, N) (for such points the exponent is not defined).

The aim of the article is the calculation of the function

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1 The physics literature devoted to the Ising model is very extensive. (The bibliography, by no means exhaustive, in M. Fisher's book [19] gives 49 titles.) The list at the end of this paper contains works which, in my opinion, may be of interest to the mathematician.

2 The existence of operators with these properties is established later in the text.

$$\Phi(\beta) = \lim_{N\to\infty} \frac{1}{N^2} \ln \Xi_N(\beta),$$

which is called the thermodynamic potential.

We mention briefly certain fundamental propositions in the statistical mechanics of a lattice gas.[1] As above, let R_N be a part of an integral lattice enclosed within a cube with edges of length $N-1$, parallel to the coordinate axes and with vertices at lattice points. Let the dimensions of the lattice be $k \geqslant 1$. The particles of the gas can be found only at the lattice points and at each point not more than one particle.

Let us examine a distribution of the particles in R_N. Suppose that the particles occupy a certain subset $M \subset R_N$. Henceforth we denote such a distribution by ξ_M. We form a real linear space by the formal linear combination of distributions ξ_M. Therefore there is a one-to-one correspondence between the subsets and the distributions of the particles, $\dim E_N = 2^{N^k}$. We examine in E_N the family of operators $n(x)$ defined in the following way:

$$n(x)\,\xi_M = \chi_M(x)\,\xi_M, \tag{3}$$

where $\chi_M(x)$ is the characteristic function of M, $n(x)$ is called the operator of the number of particles at x. From (3) it follows that the $n(x)$ have the properties:

$$n^2(x) = n(x), \qquad n(x)\,n(y) = n(y)\,n(x), \qquad \operatorname{sp} n(x) = \frac{1}{2}\dim E_N. \tag{4}$$

In accordance with the basic principles of classical statistical mechanics the probability of a distribution ξ_M is given by Gibbs' formula

$$P(M) = \frac{e^{-\frac{\beta}{2}\sum\limits_{x,\,y\in M} v(x-y)+\beta\mu N(M)}}{\sum\limits_{M} e^{-\frac{\beta}{2}\sum\limits_{x,\,y\in M} v(x-y)+\beta\mu N(M)}}, \tag{5}$$

where $N(M)$ is the number of points in the subset M. $P(M)$ is the eigenvalue corresponding to the eigenvector ξ_M of the operator

$$\frac{1}{\Xi_N} e^{-\beta\left[\sum\limits_{x,\,y\in R_N} \frac{1}{2} v(x-y)n(x)n(y) - \mu \sum\limits_{x\in R_N} n(x)\right]} \tag{6}$$

The normalizing factor Ξ_N in the denominator of (5) and (6) is called the statistical sum. It may be written in the form

$$\Xi_N = \Xi_N(\beta,\,\mu) = \operatorname{sp}\exp\left\{-\beta \sum_{x,\,y\in R_N} \frac{1}{2} v(x-y)\,n(x)\,n(y) - \mu \sum_{x\in R_N} n(x)\right\}. \tag{7}$$

The parameter β is equal to $\frac{1}{kT}$, where k is Boltzmann's constant and T is the absolute temperature, the parameter μ is called the chemical potential.

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[1] What is added below, to the end of this section, is put in to make the exposition complete, but is not essential for an understanding of the basic text of the article.

Concerning the function $v(x)$ it is assumed that $v(0) = 0$, $v(x) = v(-x)$; $v(x-y)$ has the physical meaning of the potential energy of the reciprocal action between the particles at the points x and y.

A system in which the particles interact solely with those particles that adjoin them is said to be a (general) Ising model. In other words, the Ising model is characterized by the fact that $v(x) = 0$ when $|x| > 1$ $\left(|x| = \sqrt{\sum x_i^2}\right.$, where the x_i are the integral coordinates of x).

Of physical interest are the limits, as $N \to \infty$, of the various mean physical quantities, in accordance with Gibbs' distribution (5). These limits are called thermodynamic quantities, and the limit itself is also often called the thermodynamic limit.

From (5) it follows that the thermodynamic quantities are expressed directly in terms of the statistical sum or, more exactly, the thermodynamic potential

$$\Phi(\beta, \mu) = \lim_{N\to\infty} \frac{1}{N^k} \ln \Xi_N(\beta, \mu).$$

For example, the mean density of the particles is equal to

$$\rho_N(\beta, \mu) = \sum P(M) \frac{N(M)}{N^k} = \frac{1}{N^k} \frac{1}{\beta} \frac{\partial}{\partial\mu} \ln \Xi_N.$$

In the limit $N \to \infty$:

$$\rho(\beta, \mu) = \frac{1}{\beta} \frac{\partial}{\partial\mu} \Phi(\beta, \mu).$$

Similarly, in the thermodynamic limit the mean (potential) energy is equal to $E = -\frac{\partial\Phi}{\partial\beta}$.

For statistical mechanics investigation of the singularities of the thermodynamic potential in β and μ is of primary interest. The behaviour of the system under phase transitions is bound up with these singularities. So far such an investigation has only been made of one physically interesting model (and then only partially: for β with μ fixed) - that is, for the plane Ising model with which this paper is concerned.[1]

There is a connection between the probability $P(M)$ of M and the probability of the complementary set $\bar{M}$. To show this in the form most convenient for us, let us examine the operators connected with $n(x)$ by the relation[2]

$$\sigma(x) = 2n(x) - 1. \tag{8}$$

We observe that the eigenvalues of $\sigma(x)$ are ± 1 and that if $\sigma(x)\xi_M = \xi_M$, then $\sigma(x)\xi_{\bar{M}} = -\xi_{\bar{M}}$.

Expressing $n(x)$ in terms of $\sigma(x)$ and substituting this expression in (5) and

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1 The thermodynamic potential for an Ising model is a simple function of β having one singular point. The investigation of this singularity is a simple exercise in analysis and is omitted here. We note that there are no papers in which the singularity of the thermodynamic potential in an Ising model is investigated without a preparatory *calculation* of the thermodynamic potential.

2 From (8) and (4) it follows that the operators $\sigma(x)$ satisfy equations (1) (See the note on p. 1).

(6), we find a new expression for the statistical sum and for the probability of M:[1]

$$\Xi_N = e^{N^k \beta\left(\sum_{x \in R_N} \frac{v(x)}{8} + \frac{\mu}{2}\right)} \Xi_N^I,$$

where

$$\Xi_N^I = \Xi_N^I(\beta,\ h) = \mathrm{sp} \exp \left\{ -\beta \Big[\sum_{x,\ y \in R_N} \frac{1}{8} v(x-y)\, \sigma(x)\, \sigma(y) - h \sum_{x \in R_N} \sigma(x) \Big] \right\}, \qquad (9)$$

$$h = \frac{\mu}{2} + \sum v(x).$$

The operator (6) whose eigenvalues are the probabilities $P(M)$, written in terms of $\sigma(x)$, has the form

$$\frac{1}{\Xi_N^I} e^{-\beta\left[\sum \frac{1}{8} v(x-y)\sigma(x)\sigma(y) - h \sum \sigma(x)\right]}$$

From this it follows that the Gibbs probabilities of M and $\bar{M}$ are connected by the relation $p(M,\ \beta,\ h) = p(\bar{M},\ \beta,\ -h)$. In particular, when $h = 0$ these probabilities are equal. In this case it can be shown that the system is completely invariant as far as the substitution of particles for holes or holes for particles is concerned.

It is precisely for $h = 0$ that the statistical sum in a plane Ising model is calculated. It is appropriate to note here that in the case of attraction between the particles ($v(x) \geqslant 0$) phase transition in accordance with the theorem of Lee and Yang [16] is possible only when $h = 0$.

We note, in conclusion, that another physical system, namely the system of reciprocal spins (the simple ferromagnetic model) has the same properties as the lattice gas. Under the spin interpretation of lattice-statistical physics the operators $\sigma(x)$ and the parameter h assume an independent statistical meaning: $\sigma(x)$ is the spin operator at x, and h is the rate of stress of the external magnetic field. The statistical sum in this case is given by (9), which differs from (7) by an unimportant factor. Formula (2) is a particular case(9) when $h = 0$, $k = 2$,

$$\frac{1}{4} v(\pm 1,\ 0) = -v_1, \qquad \frac{1}{4} v(0,\ \pm 1) = -v_2, \qquad v(x) = 0$$

where $x \neq (\pm 1,\ 0)$, $x \neq (0,\ \pm 1)$.

2. **Integration with respect to anti-commuting variables.**[2] Let $\mathfrak{G}$ be the complex Grassman algebra with the generators[3] $x_1,\ \ldots,\ x_n$. Functions in anticommuting variables

$$f(x) = \sum_{k \geqslant 0} \sum_{i_1 < i_2 < \ldots < i_k} f_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k}.$$

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1 Strictly speaking, the formulae given below are true only for periodic boundary conditions (that is, if we identify the opposite sides of the cube and so view the lattice not really on a cube but on a torus). In the general case there arises a distortion connected with the boundaries of the cube. It is, however, well known that this distortion is negligible for the calculation of the thermodynamic potential (see, for example, [15]).

2 The material in small print at the end of this section is put in for completeness, but is not essential for an understanding of the basic text of the article.

3 We recall that the Grassman (or exterior) algebra is defined by the sole relations between the generators: $x_i x_j + x_j x_i = 0$, $x_i^2 = 0$.

are called elements of $\mathfrak{G}$. The integral is defined in the following way:

$$\int x_i\,dx_i = 1, \qquad \int dx_i = 0.$$

A multiple integral is defined as repeated, and it is assumed that the symbols dx_i anticommute with the x_i and with each other. From this definition it follows immediately that

$$\int f(x)\,dx_n \ldots dx_1 = f_{1,\ldots,n}.$$

The integral over anti-commuting variables has many properties analogous to the properties of an ordinary integral. For us the following properties are important:

1) The formula for a linear change of variables. Let $x_i = \sum a_{ik} y_k$ where the a_{ik} are complex numbers, det $\| a_{ik} \| \neq 0$. In this case the y_i, like the x_i, are a system of generators of $\mathfrak{G}$. The following identity[1] holds:

$$\int f(x(y))\,dy_n \ldots dy_1 = \int f(x)\,dx_n \ldots dx_1 \det \| a_{ik} \|. \tag{10}$$

2) Let $\mathfrak{G}$ be an algebra with $2n$ generators $x_1, \ldots, x_n, x_1^*, \ldots, x_n^*$, Then

$$\int x_{i_1} \ldots x_{i_p} x_{i_q'}^* \ldots x_{i_1'}^* e^{\sum x_i x_i^*}\,dx_1^*\,dx_1 \ldots dx_n^*\,dx_n = \delta_{pq}(\delta_{i_1 i_1'} \ldots \delta_{i_p i_p'} \pm \ldots),$$

where the dots represent terms obtained from the first by all possible permutations of the indices $i_1', \ldots i_p'$, the sign depending on the parity of the permutation. In particular, if the set of indices $i_1, \ldots i_p$ does not coincide with $i_1', \ldots i_p'$, then the integral vanishes.

3) If a_{ik} is a skew-symmetric matrix, then[2]

$$\int e^{\frac{1}{2}\sum a_{ik} x_i x_k}\,dx_n \ldots dx_1 = (\det \| a_{ik} \|)^{\frac{1}{2}}. \tag{11}$$

Properties 1) and 2) are direct consequences of the definition of the integral, property 3) follows from 1). First let $A = \| a_{ik} \|$, a real skew-symmetric matrix. Then there exists an orthogonal unimodular matrix C such that

$$CAC^{-1} = \begin{pmatrix} \lambda_1\tau & & 0 \\ 0 & \lambda_2\tau & \\ & & \ddots \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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1 We emphasize that the analogous formula for an ordinary integral differs from (10) by the replacement of det $\| a_{ik} \|$ by $\| \det a_{ik} \|^{-1}$.

2 Concerning the sign of the square root, see below. We emphasize that the analogous formula for the ordinary integral differs (11) by the replacement of $(\det \| a_{ik} \|)^{\frac{1}{2}}$ by $\det [(2\pi)^{-1} \| a_{ik} \|]^{-\frac{1}{2}}$.

When we perform the linear substitution of variables with the matrix C, we find that

$$\int e^{\frac{1}{2}\sum a_{ik}x_ix_k}\,dx_n\ldots dx_1=\int e^{\lambda_1y_1y_2+\lambda_2y_3y_4+\cdots}\,dy_n\ldots dy_1=$$

$$=\int e^{\lambda_1y_1y_2}\,dy_2\,dy_1\int e^{\lambda_2y_3y_4}\,dy_4\,dy_3=\begin{cases}\lambda_1\ldots\lambda_{\frac{n}{2}} & \text{for even } n\\ 0 & \text{for odd } n\end{cases}=(\det\|a_{ik}\|)^{\frac{1}{2}}.$$

We now note that in accordance with the definition of the integral, the left-hand side of (6) is a polynomial in the elements of the matrix $\| a_{ik} \|$. Consequently the right-hand side also has the same property.[1] Therefore (11) is true not only for real, but also for complex matrices $\| a_{ik} \|$. Finally, the sign for the square root in (11) must be chosen in such a way that the polynomial on the right-hand side becomes $+1$ for the matrix

$$A=\begin{pmatrix}\tau & & & 0\\ & \tau & & \\ & & \ddots & \\ 0 & & & \tau\end{pmatrix},$$

where $\tau=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

We shall see below that the calculation of the statistical sum in the plane Ising model leads to an integral of type (11) with a special matrix $\| a_{ik} \|$. For the actual calculation of the integral it will be convenient to use not the final formula (11) but the method of its derivation: a linear change of variables simplifying the matrix in the exponent.

Apart from integration over a Grassman algebra, we can also introduce differentiation, in fact, not just one but two: left and right. Differential and integral calculus on a Grassman algebra is strikingly like ordinary analysis. This analogy appears already in the formulae quoted above. We give the necessary definitions and deduce some further characteristic formulae.

DEFINITIONS. a) The left-hand derivative is the linear operator in $\mathfrak{G}$ which, acting on the products of the generators, is equal to

$$\frac{\partial}{\partial x_k}x_{i_1}\ldots x_{i_p}=\delta_{k,i_1}x_{i_2}\ldots x_{i_p}-\delta_{k,i_2}x_{i_1}x_{i_3}\ldots x_{i_p}+\ldots,$$

where δ_{ij} is the Kroneker symbol. The right-hand derivative is defined similarly:

$$x_{i_1}\ldots x_{i_p}\frac{\partial}{\partial x_k}=x_{i_1}\ldots x_{i_{p-1}}\delta_{k,i_p}-x_{i_1}\ldots x_{i_{p-2}}x_{i_p}\delta_{k,i_{p-1}}+\ldots$$

b) An element $f(x)\in\mathfrak{G}$ is said to be even if it is a linear combination of products of an even number of generators, and odd if it is a linear combination

[1] Thus, in the course of events we have obtained a proof of the well-known property of skew-symmetric matrices: if $a_{ik}=-a_{ki}(i=1,\ldots,n)$ then $\det\| a_{ik} \| = R^2$, where R is a certain polynomial in the a_{ik}, and $R=0$ if n is an odd number.

of the products of an odd number of generators. If f is an even element, then $\frac{\partial}{\partial x_k} f = -f \frac{\partial}{\partial x_k}$, if f is an odd element then $\frac{\partial}{\partial x_k} f = f \frac{\partial}{\partial x_k}$. Formulae:

a) If f is an even element, then for any g

$$\frac{\partial}{\partial x_i}(fg) = \left(\frac{\partial}{\partial x_i} f\right) g + f\left(\frac{\partial}{\partial x_i} g\right),$$

$$(gf)\frac{\partial}{\partial x_i} = g\left(f \frac{\partial}{\partial x_i}\right) + \left(g \frac{\partial}{\partial x_i}\right) f.$$

If f is an odd element, then for any g

$$\frac{\partial}{\partial x_i}(fg) = \left(\frac{\partial}{\partial x_i} f\right) g - f\left(\frac{\partial}{\partial x_i} g\right),$$

$$(gf)\frac{\partial}{\partial x_i} = g\left(f \frac{\partial}{\partial x_i}\right) - \left(g \frac{\partial}{\partial x_i}\right) f.$$

b) The formula for integration by parts is true:

$$\int \left(f_1 \frac{\partial}{\partial x_i}\right) f_2 \, dx_n \ldots dx_1 = \int f_1 \left(\frac{\partial}{\partial x_i} f_2\right) dx_n \ldots dx_1.$$

c) The general formula for a change of variables is true. If

$$x_i = \sum a_{ik} y_k + \sum_{p=1} b^i_{k_1, \ldots, k_{2p+1}} y_{k_1} \cdots y_{k_{2p+1}},$$
$$\det \| a_{ik} \| \neq 0,$$

then

$$\int f(x_1(y), \ldots, x_n(y)) \Delta^{-1}(x/y) \, dy_n \ldots dy_1 = \int f(x_1, \ldots, x_n) \, dx_n \ldots dx_1,$$

where

$$\Delta(x/y) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \ldots & \frac{\partial x_1}{\partial y_n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial x_n}{\partial y_1} & \ldots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

(In the expression for the Jacobian the derivatives are written in the customary way in view of the fact that the elements x_i are odd, and for them the left- and right-hand sides coincide.)

Note that in the ordinary analysis, in the formula analogous to (6), the Jacobian has the exponent + 1.

The condition $\det \| a_{ik} \| \neq 0$ guarantees that the change of variables is invertible.

Analysis in a Grassman algebra is worked out in [17] in connection with the needs of the method of second quantification. The formula for a general change of variables is given in [18].

3. The combinatorial problem. We transform the expression (2) for the statistical sum. For brevity we put $\beta v_i = u_i$ and note that

$$e^{c\sigma(x)\sigma(y)} = I \operatorname{ch} c + \sigma(x)\sigma(y) \operatorname{sh} c = \operatorname{ch} c \,(I + \sigma(x)\sigma(y) \operatorname{th} c).$$

Using this identity we obtain for $\Xi_N(\beta)$ the new expression

$$\left.\begin{aligned}&\Xi_N(\beta)=2^{N^2}(\operatorname{ch}u_1\operatorname{ch}u_2)^{N^2}\Psi_N(\lambda_1,\lambda_2),\quad \lambda_i=\operatorname{th}u_i,\\&\Psi_N(\lambda_1,\lambda_2)=\\&=2^{-N^2}\operatorname{sp}\prod_{(m,n)\in R_N}[I+\lambda_1\sigma(m,n)\sigma(m+1,n)][I+\lambda_2\sigma(m,n)\sigma(m,n+1)].\end{aligned}\right\}\quad(12)$$

The function $\Psi_N(\lambda_1, \lambda_2)$ is a polynomial of degree $\leqslant N^2$ in λ_1 and λ_2, separately:

$$\Psi_N(\lambda_1,\lambda_2)=\sum_{A_1,A_2=0}^{N2} C_N(A_1,A_2)\lambda_1^{A_1}\lambda_2^{A_2}. \tag{13}$$

The coefficients $C_N(A_1, A_2)$ have a simple combinatorial meaning.

Let us examine all closed paths on the lattice subject to the sole condition that each lattice point belonging to a path is common to either two or four links of the path. A typical example is shown in Figure 1.

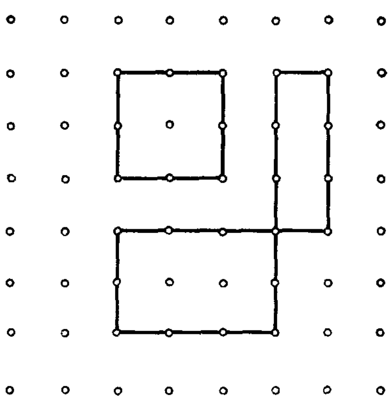

Fig. 1

$C_N(A_1, A_2)$ is the number of such paths having A_1 horizontal and A_2 vertical links. The combinatorial meaning of the coefficients $C_N(A_1, A_2)$ follows easily from (1).[1] This was first discovered by Kac and Ward [8] (see also [11]). The combinatorial problem so arising will next be solved by means of an integral over anticommuting variables.

4. **The integral representation of the function $\Psi_N(\lambda_1, \lambda_2)$ by means of an integral over anticommuting variables.** We consider the anticommuting variables [$a(m, n)$, $a^*(m, n)$, $b(m, n)$, $b^*(m, n)$, $(m, n) \in R_N$.] The variables $a(m, n)$, $a^*(m, n)$ and $b(m, n)$, $b^*(m, n)$ will henceforth be called conjugate to each other.

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[1] Expanding the expression (12) for $\Psi_N(\lambda_1, \lambda_2)$ in powers of λ_1, λ_2 we obtain a sum of terms of the form

$$2^{-N^2}\operatorname{sp}[\sigma(x_1)\ldots\sigma(x_{2A_1+2A_2})]\lambda_1^{A_1}\lambda_2^{A_2}.$$

From (1) it follows that $2^{-N^2}\operatorname{sp}[\sigma(x_1)\ldots\sigma(x_{2A_1+2A_2})]=1$ when each operator $\sigma(x_k)$ under the trace symbol occurs an even number of times (0,2 or 4 times), and that $\operatorname{sp}[\sigma(x_1)\ldots\sigma(x_{2A_1+2A_2})]=0$ otherwise. Thus, the coefficient of $\lambda_1^{A_1}\lambda_2^{A_2}$ in this expansion is different from 0 (and equal to 1) if and only if the points $x_1, \ldots, x_{2A_1+2A_2}$ lie along a contour described above. It is obvious that for this contour there are A_1 horizontal and A_2 vertical links.

The central point in the calculations is the proof of the formula

$$\Psi_N(\lambda_1, \lambda_2) = \int \prod [1 + \lambda_1\lambda_2 a^*(m-1, n)\, b^*(m, n-1) + a(m, n)\, b(m, n) + (\lambda_1 a^*(m-1, n) + \lambda_2 b^*(m, n-1))\,(a(m, n) + b(m, n)) + \lambda_1\lambda_2 a(m, n)\, b(m, n)\, a^*(m-1, n)\, b^*(m, n-1)] \times \exp\left\{\sum [a(m, n)\, a^*(m, n) + b(m, n)\, b^*(m, n)]\right\} \times \prod da^*(m, n)\, da(m, n)\, db^*(m, n)\, db(m, n). \tag{14}$$

The product in (14) is extended over all lattice points of R_N; the sum in the exponent is also taken over all points of R_N.

The subsequent calculations proceed in the following way.

We denote the contents of the square brackets in (14) by $1 + F_{m,n}$. It is easy to check that

$$1 + F_{m,n} = \exp\{\lambda_1\lambda_2 a^*(m-1, n)\, b^*(m, n) + a(m, n)\, b(m, n) + (\lambda_1 a^*(m-1, n) + \lambda_2 b^*(m, n-1))\,(a(m, n) + b(m, n))\}.$$

Therefore, the expression (14) can be rewritten in the form

$$\Psi_N(\lambda_1, \lambda_2) = \int e^{\sum[\lambda_1\lambda_2 a^*(m-1, n)\, b^*(m, n-1) + a(m, n)\, b(m, n)]} \times e^{\sum[(\lambda_1 a^*(m-1, n) + \lambda_2 b^*(m, n-1))\,(a(m, n) + b(m, n))]} \times e^{\sum[a(m, n)\, a^*(m, n) + b(m, n)\, b^*(m, n)]} \prod da^*(m, n)\, da(m, n)\, db^*(m, n)\, db(m, n). \tag{15}$$

The integral (15) is a particular case of (11). It (or rather a somewhat modified integral $\tilde{\Psi}$) is calculated in the following section.

We now turn to the plan outlined. We begin with the proof of (14).

The expression in square brackets in (14) has the form $1 + F_{m,n}$, where

$$F_{m,n} = F(a^*(m-1, n),\ b^*(m, n-1),\ a(m, n),\ b(m, n)).$$

Expanding the product we obtain a sum of expressions of the form

$$F_{m_1, n_1} \ldots F_{m_\alpha, n_\alpha}. \tag{16}$$

Now we substitute in (16) for the functions $F_{m_i n_i}$, their expressions in terms of the variables. As a result we get a sum of the form

$$\overset{(m_1, n_1)}{[\lambda_1\lambda_2 a^*(m_1-1, n_1)\, b^*(m_1, n_1-1)]}\ \overset{(m_2, n_2)}{[a(m_2, n_2)\, b(m_2 n_2)]} \ldots, \tag{17}$$

where the number pairs (m_i, n_i) over the brackets are the indices of that function, F_{m_i, n_i}, in which the expression in the brackets occurs as summand. Let us suppose that the integral of the product of (17) and $e^{\sum[aa^* + bb^*]}$ is different from zero. We mark the lattice points whose coordinates are written over the brackets in (17), and about each marked point we write the content of the corresponding bracket. Now we observe that by §2, the integral of the product of (17) and $e^{\sum[aa^*+bb^*]}$ is different from zero if and only if each variable occurs in (17) along with its conjugate. The function $F_{m,n}$ depends on $a(m, n)$, $b(m, n)$, $a^*(m-1, n)$, $b^*(m, n-1)$. Therefore, if at the point (m', n') we write $a(m', n')$, then the conjugate variable $a^*(m', n')$ is written at the point $(m'+1, n')$, next to (m', n') and situated to its right. If at the point (m'', n'') the variable $b(m'', n'')$ is written, then the conjugate variable $b^*(m'', n'')$ is written at the point $(m'', n''+1)$ next to (m'', n'') and above it. From this it follows that if the product of two variables is written at any point, then among the points in question there are two neighbouring points, and if the product is of four variables, then among the points in question there are four neighbouring points. If at one of the marked points there is written a variable $a^*(m, n)$ or $b^*(m, n)$ and at another the conjugate variable $a(m, n)$ or $b(m, n)$, then we join these two points by an arrow directed from the point without the asterisk to the point with the asterisk. (Figure 2)

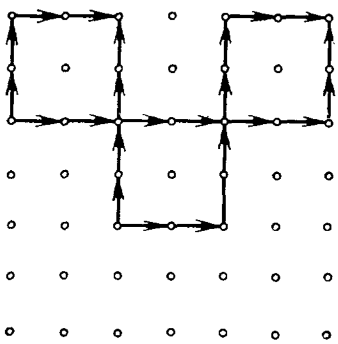

Fig. 2

It is clear from this that we turn the set of marked points into a closed path whose horizontal links go from left to right and whose vertical links go from bottom to top. The set of closed paths so obtained is the same as that which arose in the description of the statistical sum: each point on the path is joined to two or four neighbouring points of the path. It is obvious that this correspondence between expressions of the type (17) giving a non-zero contribution to the integral and the closed paths we are interested in is one to one.

We examine in more detail the contribution to the integral (13) of various closed paths. Suppose that (m, n) is a point of self-intersection. In this case we have written down the expression

$$\lambda_1\lambda_2 a^*(m-1, n)\, b^*(m, n-1)\, a(m, n)\, b(m, n).$$

We replace the point (m, n) by two neighbouring points, and about the first we write $\lambda_1\lambda_2 a^*(m-1, n)\, b^*(m, n-1)$, about the second $a(m, n)\, b(m, n)$. We join the first point to $(m-1, n)$ and $(m, n-1)$, and the second to $(m+1, n)$ and $(m, n+1)$. We leave the direction of the arrows as before. In this way for each point of self-intersection we transform the initial path into a path without self-intersections. If both points obtained as a result of the transformation of the point of self-intersection belong to the connecting component of the new path, then we call this pair of points a double point (see Figure 3. The path shown in

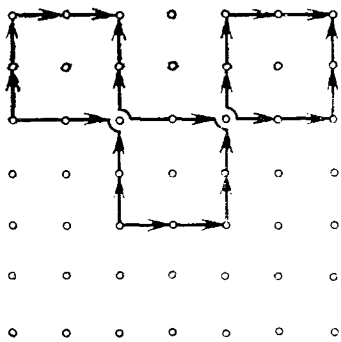

Fig. 3

this figure is obtained from that in Figure 2. It is clear that the first path has split into two connected components of which one has a double point.)

With the path so obtained we associate an expression of type (17), multiplying the factors written down for each point. Since the factors at the points where there was no self-intersection remain unaltered, and since each point of self-intersection breaks into two, while the product of the factors that correspond to the branch points is equal to the factor earlier ascribed to the original point, we see that the product of all the factors corresponding to the initial path and to the new path without self-intersections is one and the same. We may, therefore, limit ourselves to an investigation of paths without self-intersections but containing, generally speaking, double points. The possible presence of double points is not further mentioned.

Next, if the path is not connected, then the contribution from it is equal to the product of the contributions from the connected components. So we need only investigate connected closed paths without self-intersection.

We fix a point occurring in a path. The following possible cases arise: either at this point one link begins and one ends, or two links end, or two links begin. At the point the product of a numerical factor and of two anticommuting variables is written. In the first case we attribute the numerical factor to that link which ends at the given point. In the second case the numerical factor is equal to $\lambda_1\lambda_2$, one of the links is horizontal and the other vertical. We give factor λ_1 to the horizontal link, and λ_2 to the vertical. Finally, in the third case the numeral factor is equal to 1. Thus, for each horizontal link in the path we put λ_1, for each vertical, λ_2. By §2 the integral over the product of the anticommuting factors assigned to the vertices of the path and $e^{\sum[aa^*+bb^*]}$ is equal to $+1$ or -1. Consequently, the contribution from a closed path without self-intersection, to within a sign, is equal to $\lambda_1^{A_1}\lambda_2^{A_2}$, where A_1 is the number of horizontal and A_2 the number of vertical links in the path. We show that the contribution, in fact, is +1.

We call a point regular if one of its relevant links begins and the other ends at it. We call the remaining points singular. We say that a singular point is of the first kind if both its links begin, and of the second kind if they both end at it.

Let us examine a singular point of the first kind (m_0, n_0) and beginning with it go around the contour moving first in a vertical direction. We write out in the order in which we meet them the variables of integration

$$[a(m_0, n_0)\, b(m_0, n_0)]\, [b^*(m_0, n_0)\, b(m_0, n_0+1)]\, [b^*(m_0, n_0+1)\, a(m_0, n_0+2)] \ldots \tag{18}$$

We begin to integrate the product (18) with the Gaussian weight factor $e^{\sum[aa^*+bb^*]}$, moving from left to right: first we integrate $b(m_0, n_0)\, b^*(m_0, n_0) \times \times e^{b(m_0, n_0)\, b^*(m_0, n_0)}$, then $b(m_0, n_0+1)\, b^*(m_0, n_0+1)\, e^{b(m_0, n_0+1)\, b^*(m_0, n_0+1)}$ and so on. It is easy to see that as long as we go in the direction indicated by the arrows, that is, as we do not reach the next singular point, we get +1 on integration. The first singular point we reach is obviously a point of the second kind. The last link before this point may be either horizontal or vertical. We look at the two cases separately.

LAST LINK HORIZONTAL. The part of the product (18) corresponding to the immediate neighbourhood of the singular point (m_1, n_1) has the form

$$\ldots [\varepsilon^* a(m_1-1, n_1)]\, [a^*(m_1-1, n_1)\, b^*(m_1, n_1-1)]\, [\sigma^* b(m_1, n_1-1)] \ldots$$

(Here ε^* denotes one of the possible variables preceding $a(m_1-1, n_1)$: either $\varepsilon^* = a^*(m_1-2, n_1)$, or $\varepsilon^* = b^*(m_1-1, n_1-1)$. Similarly $\sigma^* = b^*(m_1, n_1-2)$, or $\sigma^* = a^*(m_1, n_1-1)$.)

The process described above ends with the integration of the expression $\varepsilon\varepsilon^* e^{\varepsilon\varepsilon^*}$. We continue this process, integrating $a(m_1-1, n_1)\, a^*(m_1-1, n_1) \times e^{a(m_1-1, n_1)\, a^*(m_1-1, n_1)}$. As a result we obtain a product of the form

$$a(m_0, n_0)\, b^*(m_1, n_1-1)\, [\sigma^* b(m_1, n_1-1)]\, [\mu^* \sigma] \ldots, \tag{19}$$

where μ^* denotes the variable occurring in the pair with σ. Before integrating (19) we interchange $b^*(m_1, n_1-1)$ with the two variables to its right:

$$a(m_0, n_0)\, [\sigma^* b(m_1, n_1-1)]\, b^*(m_1, n_1-1)\, [\mu^* \sigma] \ldots \tag{20}$$

The products (19) and (20) are equal. Integrating

$$b(m_1, n_1-1)\, b^*(m_1, n_1-1)\, e^{b(m_1, n_1-1)\, b^*(m_1, n_1-1)},$$

we get from (20) an expression of the same structure as (19):

$$a(m_0, n_0)\,\sigma^*\,[\mu^*\sigma]\ldots$$

Consequently we can apply the preceding method: interchange σ^* to the right of the bracket and integrate $\sigma\sigma^* e^{\sigma\sigma\nu}$. Continuing the process further we come to integrals corresponding to the next singular point. We mention that on this part of the path, when we move along the contour in the direction opposite to that indicated by the arrows, at each step we get an expression of type (19).

Now we examine the second possible approach to the first singular point.

LAST LINK VERTICAL. In this case the part of the product (18) corresponding to the neighbourhood of the singular point (m_1, n_1) has the form

$$\ldots[\varepsilon^* b(m_1, n_1-1)]\,[a^*(m_1-1, n_1)\,b^*(m_1, n_1-1)]\,[\sigma^* a(m_1-1, n_1)]\ldots$$

We shall integrate over the variables preceding $b(m_1, n_1-1)$, and interchange $a^*(m_1-1, n_1)$ and $b^*(m_1, n_1-1)$. Since the variables anticommute, we get a factor (-1). Integrating then

$$b(m_1, n_1-1)\,b^*(m_1, n_1-1)\,e^{b(m_1, n_1-1)\,b^*(m_1, n_1-1)},$$

we obtain an expression of type (19):

$$a(m_0, n_0)\,\sigma^*\,[\mu^*\sigma]\ldots$$

Applying the preceding arguments we find that as long as we don't reach the variables corresponding to the next singular point, all the following integrals are +1.

The next singular point, like the initial point, is of the first kind. As we come up to it we must again take separately the case when the last link is vertical, and that when it is horizontal.

LAST LINK VERTICAL. That part of the product (18) corresponding to the neighbourhood of the singular point (m_2, n_2) has the form

$$\ldots[b^*(m_2, n_2)\,\varepsilon]\,[a(m_2, n_2)\,b(m_2, n_2)]\,[a^*(m_2, n_2)\,\sigma]\ldots$$

After integrating over all variables up to and including ε, ε^* we get

$$\varkappa a(m_0, n_0)\,b^*(m_2, n_2)\,[a(m_2, n_2)\,b(m_2, n_2)]\,[a^*(m_2, n_2)\,\sigma]\ldots,$$

where $\varkappa = +1$ or $\varkappa = -1$, depending on whether the last link before (m_1, n_1) is horizontal or vertical. We shift $b^*(m_2, n_2)$ to the right of the first bracket written down and integrate

$$b(m_2, n_2)\,b^*(m_2, n_2)\,e^{b(m_2, n_2)\,b^*(m_2, n_2)}.$$

As a result we get an expression that differs from (18) only by a factor and by the notation of the variables:

$$\varkappa a(m_0, n_0)\,a(m_2, n_2)\,(a^*(m_2, n_2)\,\sigma)\ldots \tag{21}$$

Thus, we return to the situation already reviewed. Now we pass on to the second case.

LAST LINK HORIZONTAL. The part of product (18) corresponding to the neighbourhood of the singular point has in this case the form

$$\ldots [a^* (m_2, n_2)\, \varepsilon]\, [a (m_2, n_2)\, b (m_2, n_2)]\, [b^* (m_2, n_2)\, \sigma] \ldots$$

After integrating over all variables up to and including ε, ε^* we get

$$\varkappa a (m_0, n_0)\, a^* (m_2, n_2)\, [a (m_2, n_2)\, b (m_2, n_2)]\, [b^* (m_2, n_2)\, \sigma] \ldots$$

We note that $\int a^* a e^{aa^*}\, da^*\, da = -1$. Therefore, integration over $a^*(m_2, n_2)$, $a(m_2, n_2)$ gives

$$(-1)\, \varkappa a (m_0, n_0)\, b (m_2, n_2)\, [b^* (m_2, n_2)\, \sigma] \ldots$$

The remaining product differs from (21) only by the notation of the variables. Any subsequent movement along the contour repeats these situations.

The last singular point on our path is of the second kind. Therefore in the last part of the path, consisting of regular points the integrand has the form (19). Omitting the sign we write down the integrand corresponding to the immediate neighbourhood of the initial point (m_0, n_0):

$$a (m_0, n_0)\, \sigma^* \; [a^* (m_0, n_0)\, \sigma]. \qquad (22)$$

Transferring σ^* to the right we see that the integral of the product of (22) and the Gaussian factor $e^{a(m_0, n_0)\, a^* (m_0, n_0) + \sigma\sigma^*}$ is $+1$.

We denote by N_1 the number of singular points of the first kind that we pass on going round the contour so that the last link is horizontal, and by N_2 the number of singular points of the second kind that we pass on going round the contour so that the last link is vertical. The analysis given shows that the sign of the integral of the product (17) and the Gaussian factor is equal to $(-1)^{N_1+N_2-1}$. The term -1 in the exponent arises because the initial point plays a particular role: although on approaching it the last link is horizontal, nevertheless on passing through it the sign does not change.

Thus, our assertion concerning the sign is a consequence of the following geometrical lemma:

LEMMA. *Let $\mathcal{L}$ be a connected, closed, simple contour on a square lattice (which may have double points) in which each horizontal link goes from left to right and each vertical link from bottom to top.*

We call a point of the contour from which two links start a singular point of the first kind, and a point at which two links end a singular point of the second kind.

We examine the circuit around the contour, denote by N_1 the number of singular points of the first kind that we pass so that the last link is horizontal, and by N_2 the number of singular points of the second kind that we pass so that the last link is vertical.

Then the number $N_1 + N_2$ is odd.

The lemma is proved in §6.

Returning to the integral (15) we find that it is equal to the right-hand side of (13), which defines the function $\Psi(\lambda_1, \lambda_2)$.

5. Calculation of the thermodynamic potential. We suppose for definiteness that the initial square contains the points $x = (m, n)$ with the coordinates $0 \leqslant m$, $n \leqslant N - 1$. We add to it the points (m, N), (N, n), (N, N) and identify the opposite sides of the larger square: we put $(m, N) = (m, 0)$, $(N, n) = (0, n)$, $(N, N) = (0, 0)$. So we get an integral lattice on a torus which we denote T_N. Now we replace the quadratic form in the exponent of the integrand in (15) by a similar form in which the summation extends over all integral points of T_N.

The new quadratic form differs from the old one by the additional terms

$$\sum [\lambda_1\lambda_2 a^* (m-1, 0)\, b^* (m, N-1) + \ldots].$$

Let the matrix of the old quadratic form be A_N, and of the new $\widetilde{A}_N$ The integral obtained by substituting in (15) $\widetilde{A}_N$ for A_N is denoted by $\widetilde{\Psi}_N$. To calculate the thermodynamic potential we must find $\lim\limits_{N\to\infty} \frac{1}{N^2} \ln \Psi_N$. In accordance with (11) this limit is equal to $\frac{1}{2} \lim\limits_{N\to\infty} \frac{1}{N^2} \ln \det A_N$. Since A_N and $\widetilde{A}_N$ differ only in their boundary terms, we have $\lim\limits_{N\to\infty} \frac{1}{N^2} \ln \det A_N = \lim\limits_{N\to\infty} \frac{1}{N^2} \ln \det \widetilde{A}_N$. The matrix $\widetilde{A}_N$ is block-cyclic[1] and can therefore easily be reduced to a block-diagonal form which permits us to

[1] This means that $\widetilde{A}_N = \| a(m, n; m', n') \| = \| \alpha(m-m', n-n') \|$, where the $a(m, n)$ are matrices (in our case of order 4), and $(m, n) \in T_N$, $(m', n') \in T_N$.

find its determinant.[1]

The calculation becomes most convenient when we make a change of variables in the integral $\tilde{\Psi}_N$.

We introduce new variables by means of a Fourier transformation:

[1] In all the derivations of Onsager's formula known so far the same situation arises: by means of one reduction or another the statistical sum can be expressed in the form det $A_N^{(i)}$ (i is an index denoting the author), where $A_N^{(i)}$ is a matrix whose elements have lattice points of R_N as indices. Then by means of the addition of boundary elements $A_N^{(i)}$ becomes a block-cyclic matrix $\tilde{A}_N^{(i)}$ whose determinant can also be calculated. The limit equation

$$\lim_{N\to\infty}\frac{1}{N^2}\ln\det A_N^{(i)} = \lim_{N\to\infty}\frac{1}{N^2}\ln\det\tilde{A}_N^{(i)} \qquad (*)$$

does not worry the authors so that they do not even notice the replacement of $A_N^{(i)}$ by $\tilde{A}_N^{(i)}$ (see, for example, [11]). However, [12] is an exception, for here an attempt is made to prove the equation; but in fact all that is established is

$$\lim_{N\to\infty}\frac{1}{N^2}\ln\det A_N^{(i)} \leqslant \lim_{N\to\infty}\frac{1}{N^2}\ln\det\tilde{A}_N^{(i)}.$$

So far there are apparently no mathematical papers to which we could refer for a rigorous proof of the equation we need. Therefore, our derivation of Onsager's formula is not, strictly speaking, complete.

To support our firm opinion on the validity of the equation (*) we note that it is very easy to prove the analogous equation when $\tilde{A}_N^{(i)}$ is not a block-cyclic, but simply a cyclic matrix (that is, it has the form described in the preceding remark, but the elements $\alpha(m, n)$ are not matrices but numbers).

In conclusion we draw attention to a curious circumstance. We replace in (2) the summation over R_N by that over the torus T_N. The statistical sum so modified is denoted by $\Xi'_N(\beta)$. It is easy to prove (for example, in [15]) that

$$\lim_{N\to\infty}\frac{1}{N^2}\Xi_N(\beta) = \lim_{N\to\infty}\frac{1}{N^2}\Xi'_N(\beta).$$

The calculation of $\Xi'_N(\beta)$ reduces to a combinatorial problem just like that for the calculation of $\Xi_N(\beta)$, but with this difference that the paths now examined are not on the lattice R_N but on the torus T_N. At first glance it seems that we could then proceed exactly as for the calculation of $\Xi_N(\beta)$. It turns out, however, that the geometrical lemma on which the proof of the integral representation [14] is based is not valid for lattices on the torus!

$$\left.\begin{aligned} a(m, n) &= \frac{1}{N}\sum_{pq} \alpha(p, q)\, e^{\frac{2\pi i}{N}(mp+nq)}, \\ b(m, n) &= \frac{1}{N}\sum_{pq} \beta(p, q)\, e^{\frac{2\pi i}{N}(mp+nq)}, \\ a^*(m, n) &= \frac{1}{N}\sum_{pq} \alpha^*(p, q)\, e^{-\frac{2\pi i}{N}(mp+nq)}, \\ b^*(m, n) &= \frac{1}{N}\sum_{pq} \beta^*(p, q)\, e^{-\frac{2\pi i}{N}(mp+nq)}, \end{aligned}\right\} \tag{23}$$

where p and q are pairs of integers which we may regard as coordinates of the lattice points on the torus T.

From the relation

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{\frac{2\pi inp}{N}} = \begin{cases} 1 & \text{when} \quad p=0, \\ 0 & \text{when} \quad p \neq 0 \end{cases} \tag{24}$$

it follows that the matrix of the transformation (23) is unitary and that its determinant is 1. Using (24) it is easy to find an expression for the exponent in terms of the Greek variables:

$$\sum a^*(m-1, n)\, b^*(m, n-1) = \sum \alpha^*(p, q)\, \beta^*(-p, -q)\, e^{\frac{2\pi i}{N}(p-q)}.$$

The remaining terms are transformed similarly. As a result the exponent becomes equal to $\Sigma Q(p, q)$, where

$$Q(p, q) = \lambda_1\lambda_2\alpha^*(p, q)\, \beta^*(-p, -q)\, e^{\frac{2\pi i}{N}(p-q)} + \\ + (\lambda_1\alpha^*(p, q)\, e^{\frac{2\pi ip}{N}} + \lambda_2\beta^*(p, q)\, e^{\frac{2\pi iq}{N}})\, (\alpha(p, q) + \beta(p, q)) + \\ + \alpha(p, q)\, \alpha^*(p, q) + \beta(p, q)\, \beta^*(p, q).$$

Thus, the integral is split into the product of the integrals that correspond to the points pairs (p, q) and $(-p, -q)$, and of the integral corresponding to $(0, 0)$. The integral over the eight variables corresponding to (p, q) and $(-p, -q)$ can be easily calculated:

$$\int e^{Q(p, q) + Q(-p, -q)} \times \\ \times d\alpha^*(p, q)\, d\alpha(p, q)\, d\alpha^*(-p, -q)\, d\alpha(-p, -q)\, d\beta^*(p, q)\, d\beta(-p, -q) = \\ = 1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2 + 2\lambda_1\lambda_2\left(\lambda_1 \cos\frac{2\pi p}{N} + \lambda_2 \cos\frac{2\pi q}{N}\right) - \\ - 2\lambda_1 \cos\frac{2\pi p}{N} - 2\lambda_2 \cos\frac{2\pi q}{N}.$$

The integral over the variables $\alpha(0, 0)$, $\beta(0, 0)$, $\alpha^*(0, 0)$, $\beta^*(0, 0)$ is obtained from the expression above by putting in it $p = q = 0$ and extracting the square root. (The sign of the root must be chosen so that when $\lambda_1 = \lambda_2 = 0$, we obtain $+1$.) Thus, for the function $\tilde{\Psi}_N$ we get the final expression

$$\tilde{\Psi}_N(\lambda_1, \lambda_2) = \prod \Big[1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2 + 2\lambda_1\lambda_2\left(\lambda_2 \cos\frac{2\pi p}{N} + \lambda_1 \cos\frac{2\pi q}{N}\right) - -2\lambda_1 \cos\frac{2\pi p}{N} - 2\lambda_2 \cos\frac{2\pi q}{N}\Big]^{\frac{1}{2}}.$$

The product is taken over all lattice points on the torus T_N, the exponent $\frac{1}{2}$ arises from the fact that the factors corresponding to each pair of points (p, q), $(-p, -q)$ occur twice.

Taking into account the connection between the function $\Psi_N(\lambda_1, \lambda_2)$ and the statistical sum $\Xi_N(\beta)$, we obtain from this an expression for $\Xi_N(\beta)$. Taking the logarithm, dividing $\ln \Xi_N(\beta)$ by N^2, going to the limit as $N \to \infty$, and taking into account that $\lim \frac{1}{N^2} \ln \Psi_N = \lim \frac{1}{N^2} \ln \tilde{\Psi}_N$, we find the well-known expression for the thermodynamic potential:

$$\Phi(\beta) = \lim_{N\to\infty} \frac{1}{N_2} \ln \Xi_N(\beta) = \ln 2 + \ln(\operatorname{ch}\beta v_1 \cdot \operatorname{ch}\beta v_2) + + \frac{1}{8\pi^2} \int_0^{2\pi}\int_0^{2\pi} \ln[1 + \lambda_1^2 + \lambda_1^2 + \lambda_1^2\lambda_2^2 + 2\lambda_1\lambda_2(\lambda_1 \cos\varphi_2 + \lambda_2 \cos\varphi_1) - -2\lambda_1 \cos\varphi_1 - 2\lambda_1 \cos\varphi_2]\, d\varphi_1\, d\varphi_2,$$

where $\lambda_1 = \operatorname{th}\beta v_1$, $\lambda_2 = \operatorname{th}\beta v_2$.

A few remarks about our result. First let us suppose that $v_1 > 0$ and $v_2 > 0$ (This means that the particles attract one another.) In this case also $\lambda_1 > 0$ and $\lambda_2 > 0$. (We assume that $\beta > 0$ as a consequence of the physical meaning of the parameter.) The expression under the logarithm has a minimum when $\varphi_1 = \varphi_2 = 0$: this minimum is equal to

$$K_{++}(\beta) = (1 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2)^2;$$

It is not difficult to see that $\Phi(\beta)$ is a holomorphic analytic function in the vicinity of each point β_0 for which $K_{++}(\beta_0) > 0$, and that $\Phi(\beta)$ has a singularity at β_c, which is a root of the equation $K_{++}(\beta) = 0$. This equation has a unique root $\beta_c > 0$ which defines the temperature of the phase transition $T_c = \frac{1}{k\beta_c}$, where k is Boltzmann's constant. In the case $v_1 < 0$ and $v_2 < 0$ (repulsion), the expression under the logarithm sign has a minimum when $\varphi_1 = \varphi_2 = \pi$, and this minimum is equal to

$$K_{--}(\beta) = (1 + \lambda_1 + \lambda_2 - \lambda_1\lambda_2)^2.$$

We can treat similarly the cases $v_1 > 0$, $v_2 < 0$ and $v_1 < 0$, $v_2 > 0$. The expression under the logarithm in these cases again is strictly positive for all $\beta \neq \beta_c$ and is equal to zero for the unique value β_c. The temperature of the phase transition is defined in the same way. It is curious to note that it does not change under the substitution $v_1 \to -v_1$, $v_2 \to -v_2$.

We note that the transition from the sum of logarithms to the integral (26), which we made when $N \to \infty$, is justified because the expression under the logarithm sign in (26) is positive.

6. Proof of the lemma. We denote by $\mathscr{L}$ a connected closed simple contour on a plane lattice; we go round it and mark those singular points of the first kind which we approach so that the last link is horizontal, and those singular points of the second kind which we approach so that the last link is vertical. Let the total number of marked points be $\varkappa(\mathscr{L})$. We have to show that $\varkappa(\mathscr{L})$ is odd.

We begin with an intuitive argument. We observe first of all that instead of considering a closed, simple broken line whose links consist of horizontal and vertical segments joining the points of an integral lattice, we may consider a similar broken line consisting of horizontal and vertical links that are in no way connected with the lattice. Let $\mathscr{L}$ be a broken line of this kind, and L_1, L_2, L_3 its parts consisting, respectively, of a straight-line segment, of two straight line segments and the angle between them, and of three straight line segments and the two angles between them. We say that a contour $\mathscr{L}'$ is an elementary transformation of $\mathscr{L}$ if it is obtained from $\mathscr{L}$ by replacing one of the parts L_i by L_i', where L_i' is a broken line completing L_i to a rectangle, for example, L_1' may consist of three segments and two angles and have the same ends as L_1. It is easy to check that $\varkappa(\mathscr{L}) = \varkappa(\mathscr{L}')$ or $\varkappa(\mathscr{L}) = \varkappa(\mathscr{L}') \pm 2$ depending on the direction of the circuit. On the other hand it is clear that by a sequence of elementary transformations any closed simple contour with sides parallel to the coordinate axes can be transformed into a rectangle. Since for a rectangle $\varkappa(\mathscr{L}) = 1$ this then establishes that for any contour $\varkappa(\mathscr{L}) \equiv 1 \pmod 2$.

Unfortunately rigorous proof that an arbitrary contour can be transformed into a rectangle by means of elementary transformations is complicated, therefore a complete proof of the lemma is based on another idea.

We examine the smallest possible rectangle with sides parallel to the coordinates axes such that there are no points of the contour outside. We call its sides, respectively, the lower, upper, left and right supporting lines of the contour. We call the distance between the left and the right supporting lines the width of $\mathscr{L}$ and denote it by $s(\mathscr{L})$. We order the points of $\mathscr{L}$ lexicographically: $x_1 = (m_1, n_1) < x_2 = (m_2, n_2)$ if $m_1 < m_2$ or if $m_1 = m_2$ and $n_1 < n_2$. The least point in the sense of this order, $x_0 = (m_0, n_0)$, is called the beginning of L, the largest is $\bar{x}_0 = (\bar{m}_0, \bar{n}_0)$.

It is obvious that $s(\mathscr{L}) = \bar{m}_0 - m_0$. If $s(\mathscr{L}) = 1$, the contour is a rectangle. In that case, $\varkappa(\mathscr{L}) = 1$, and the Lemma is true. If $s(\mathscr{L}) > 1$, we construct a new contour $\mathscr{L}_1$ with the properties 1) $\varkappa(\mathscr{L}_1) \equiv \varkappa(\mathscr{L}) \bmod 2$; 2) the lower, upper and right supporting lines of $\mathscr{L}_1$ lie respectively, not lower, nor higher, nor to the right of the analogous lines for $\mathscr{L}$; 3) The beginning x_1 of $\mathscr{L}_1$ satisfies $x_1 > x_0$, where x_0 is the beginning of $\mathscr{L}$.

The proof of the Lemma is contained in the construction of the contour $\mathscr{L}_1$. For $s(\mathscr{L}_1) > 1$, then we construct a $\mathscr{L}_2$, in a similar way, and if necessary, $\mathscr{L}_3$ and so on. Since the beginnings of these contours satisfy the inequalities $x_i > x_{i-1}$ and since all the contours satisfy condition 2, there must be in this sequence a contour $\mathscr{L}_{k_1}$ of smaller width than $\mathscr{L}$. If $s(\mathscr{L}_{k_1}) > 1$, then for the same reason there is a contour $\mathscr{L}_{k_2}$ such that $s(\mathscr{L}_{k_2}) < s(\mathscr{L}_{k_1})$, and so on. Consequently in the sequence $\mathscr{L}_k$ there is a rectangle $\mathscr{L}_n$. As we have already noted, $\varkappa(\mathscr{L}_n) = 1$ and so $\varkappa(\mathscr{L}) \equiv 1 \pmod 2$.

We proceed to the construction of the contour $\mathscr{L}_1$. We move from x_0 to the right one step and produce a vertical line l. We denote by y_1 the point on l, joined to x_0 by a horizontal link; and by y_k ($k = 2, 3, \ldots$), the points on l belonging to $\mathscr{L}$ and lying above y_1, numbering from bottom to top.

We say that the point y_i for $i > 1$ is in general position if it does not lie on a vertical link of $\mathscr{L}$. We call y_i a point of general position if it is not the lower end of a vertical link.

Let us examine the possibilities that arise. (They are all illustrated in Figure 4.)

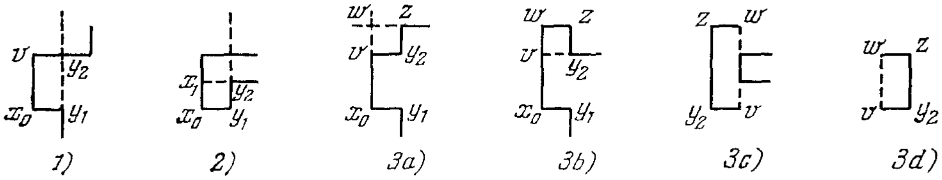

Fig. 4

1. The points y_1 and y_2 are in general position. Let v be the left end of the segment going through y_2. Next we rub out the sections (x_0, y_1), (x_0, v), (v, y_2), and introduce (y_1, y_2). As a result we get a contour $\mathscr{L}_1$ with the beginning $x_1 > x_0$, for which $\varkappa(\mathscr{L}_1) = \varkappa(\mathscr{L}_2)$, if y_1 does not lie on the vertical segment; otherwise $\varkappa(\mathscr{L}_1) = \varkappa(\mathscr{L}_2)$ or $\varkappa(\mathscr{L}_1) = \varkappa(\mathscr{L}) - 2$ depending on the direction of the circuit.

2. The point y_2 is not in general position. In this case y_1 and y_2 lie on a segment belonging to the contour. Through y_2 to the left we produce a horizontal segment until it cuts the vertical segment through x_0. Let x_1 be the point of intersection. Now we rub out the segments (x_0, x_1), (x_0, y_1), (y_1, y_2). As a result we get a new contour $\mathscr{L}_1$. It is obvious that $\varkappa(\mathscr{L}_1) = \varkappa(\mathscr{L})$.

3. The point y_1 is in general position, but y_2 is not. It is obvious that y_2 is the lower end of a segment belonging to $\mathscr{L}$. Let z be the upper end of this segment. These cases are possible:

a) a horizontal segment, with end y_2 directed to the left; a horizontal segment with end z directed to the right. Let v be the left end of that segment on which y_2 lies. From point z we produce one step to the left a segment whose end we call w. We join the points v and w and omit the segments (y_2, z) and (v, y_2). As a result we get a contour $\mathscr{L}'$ in which z in general position plays the part of y_2. It is obvious that $\varkappa(\mathscr{L}) = \varkappa(\mathscr{L}')$.

b) a horizontal segment, with end y_2 directed to the right; a horizontal segment with end z directed to the left. We call the left end of this segment w. From y_2 we produce to the left a segment of length 1 whose end we call v. It is easy to check that the point v belongs to $\mathscr{L}$. We omit the segments (y_2, z), (z, w), (v, w). In $\mathscr{L}'$, the newly obtained contour y_2 is a point in general position and, depending on the direction of the circuit, $\varkappa(\mathscr{L}') = \varkappa(\mathscr{L})$ or $\varkappa(\mathscr{L}') = \varkappa(\mathscr{L}) - 2$.

c) Both horizontal segments going through y_2 and z directed to the right. As before, let v be the point nearest to y_2 on the horizontal segment, and w the point nearest to z. We suppose to begin with that the segment (w, v) contains no points of the contour except w and v. In this case we join v and w and omit the segments (y_2, z), (y_2, v), (z, w). As a result we get a new contour $\mathscr{L}'$ in which, depending on the direction of the circuit, $\varkappa(\mathscr{L}') = \varkappa(\mathscr{L})$ or $\varkappa(\mathscr{L}') = \varkappa(\mathscr{L}) - 2$. We call the construction so described the extrusion to the right of the broken line vy_2zw.

If on the segment w, v there are points of the contour other than its ends, they occur as pairs y_2^i, z^i, similar to y_2, z. With each pair of points y_2^i, z^i there is connected a pair of points w^i, v^i similar to w, v. If on the segment w^i, v^i there are points of the contour other than w^i, v^i, then these points occur as pairs $y_2^{i,j}$, $z^{i,j}$ similar to those described above, and with them, as before, there are connected pairs $w^{i,j}$, $v^{i,j}$. Continuing further we obtain a system of broken lines $v^{i_1,\dots,i_p}$, $y_2^{i_1,\dots,i_p}$, $z^{i_1,\dots,i_p}$, $w^{i_1,\dots,i_p}$. We shall call such a broken line terminal if on the segment $w^{i_1,\dots,i_p}$ $v^{i_1,\dots,i_p}$ there are no points of the contour other than $w^{i_1,\dots,i_p}$, $v^{i_1,\dots,i_p}$. We extrude all terminal lines to the right. Repeating this construction, if necessary, we eventually obtain a contour whose segment w, v does not contain any points other than w and v.

d) Both segments going through y_2 and z are directed to the left. In this case the contour $\mathscr{L}'$ is constructed in the same way. (It is obvious that the segment (w, v) does not contain points of the contour other than v and w.)

In the cases 3c) and 3d) we denote by y_2' that point on $\mathscr{L}$ which is nearest to and above y_1 on the straight line l. If y_2' is in general position, then we reach the situation discussed with in the first variant. If y_2' is not in general position, then the situation arising is 3a), 3b), 3c) or 3d). In the cases 3a) and 3b) we construct, as described in the relevant place, a contour $\mathscr{L}''$ whose corresponding point is in general position. In the remaining cases we continue the construction as described in 3c) and 3d). Since this cannot be an infinite process, we eventually reach a contour $\tilde{\mathscr{L}}$ with a point $\tilde{y}_2$ in general position.

In this way a contour $\mathscr{L}_1$ can be constructed for all possible cases and the Lemma is proved.

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Received by the Editors,
February 17, 1969.

Translated by D.W. Jordan.